CHARACTERIZATIONS OF PARTITION LATTICES

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I. Introduction

One of the most well-known geometric lattices is a partition lattice. Every upper interval of a partition lattice is a partition lattice. The Whitney numbers of the partition lattices are the Stirling numbers, and the characteristic polynomial is a falling factorial. The set of partitions with a single non-trivial block containing a fixed element is a Boolean sublattice of modular elements, so the partition lattice is supersolvable in the sense of Stanley [6].

In this paper, we rephrase four results due to Heller [1] and Murty [4] in terms of matroids and give several characterizations of partition lattices.

Our notation and terminology follow those in [8, 9]. To clarify our terminology, let \( G \) be a finite geometric lattice. If \( S \) is the set of points (or rank-one flats) in \( G \), the lattice structure of \( G \) induces the structure of a (combinatorial) geometry, also denoted by \( G \), on \( S \). The size \(|G|\) of the geometry \( G \) is the number of points in \( G \). Let \( T \) be a subset of \( S \). The deletion of \( T \) from \( G \) is the geometry on the point set \( S \setminus T \) obtained by restricting \( G \) to the subset \( S \setminus T \). The contraction \( G/T \) of \( G \) by \( T \) is the geometry induced by the geometric lattice \([cl(T), \hat{1}]\) on the set \( S' \) of all flats in \( G \) covering \( cl(T) \). (Here, \( cl(T) \) is the closure of \( T \), and \( \hat{1} \) is the maximum of the lattice \( G \).) Thus, by definition, the contraction of a geometry is always a geometry. A geometry which can be obtained from \( G \) by deletions and contractions is called a minor of \( G \).

2. Preliminaries

Let \( S \) be a finite set of \( n \) elements. A partition \( \pi \) of \( S \) is a family of disjoint subsets \( B_1, B_2, \ldots, B_k \), called blocks, whose union is \( S \). There is a natural
ordering of partitions, which is defined as follows: \( \pi \leq \sigma \) whenever every block of a partition \( \pi \) is contained in a block of a partition \( \sigma \). Denote the lattice of partitions of a set with \( n \) elements by \( P_n \). We call \( P_n \) the \textit{partition lattice} of rank \( n - 1 \).

Let \( G \) be a geometry. Then we can associate with \( G \) a geometric lattice \( \mathcal{L}(G) \) whose elements are the flats of \( G \) ordered by inclusion. Note that the partition lattice \( P_n \) is isomorphic to the lattice of flats of the polygon matroid of the complete graph \( K_n \).

**Theorem 1** [1]. Let \( G \) be a binary geometry of rank \( n \) and let \( a \) be a point in \( G \). If \( |G| - |G/a| > n \), then \( G \) contains the Fano plane as a minor.

**Corollary 1** [1, 4]. A binary rank-\( n \) geometry not containing the Fano plane as a minor contains at most \( \binom{n+1}{2} \) points.

As a contrapositive of Corollary 1, we have the following.

**Corollary 2** [1, 4]. If a binary geometry of rank \( n \) has more than \( \binom{n+1}{2} \) points, then it contains the Fano plane as a minor.

**Theorem 2** [1, 4]. If a binary rank-\( n \) geometry \( G \) not containing the Fano plane as a minor contains \( \binom{n+1}{2} \) points, then \( G \) is the polygon matroid of the complete graph \( K_{n+1} \); that is, \( \mathcal{L}(G) \cong P_{n+1} \).

**Theorem 3.** If a geometry \( G \) has \( \binom{n+1}{2} \) points and \( \mathcal{L}(G/a) \cong P_n \) for every point \( a \) in \( G \), then \( \mathcal{L}(G) \cong P_{n+1} \).

**Proof.** Note that \( G \) has rank \( n \). Since \( \mathcal{L}(G/a) \cong P_n \) for every point \( a \) in \( G \), the scum theorem [9, p.240] implies that \( G \) is binary. For \( n = 1 \) and \( n = 2 \), the theorem is true.

Let \( n = 3 \). If \( G \) contains the Fano plane as a minor, then \( G \) is isomorphic to the Fano plane. But \( G \) and the Fano plane have a different number of points, so we have a contradiction. Thus \( G \) cannot contain the Fano plane as a minor. By Theorem 2, we have \( \mathcal{L}(G) \cong P_4 \).

Let \( n \geq 4 \). If \( G \) contains the Fano plane as a minor, then by the scum theorem \( G/a \) contains the Fano plane as a minor for some point \( a \) in \( G \). Since
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$L(G/a) \cong P_n$, the polygon matroid of the complete graph $K_n$ contains the Fano plane as a minor. But this is contradictory to [10, Theorem 1.5.4.]. Thus $G$ cannot contain the Fano plane as a minor. By Theorem 2, we have $L(G) \cong P_{n+1}$.

Kahn and Kung [3] defined splitting in geometries. Let $G$ be a geometry. Then $G$ splits if $G$ is the union of two of its proper flats. $G$ is said to be non-splitting otherwise. We shall be more concerned with non-splitting geometries. An example of a non-splitting geometry is $M(K_n)$, the polygon matroid of the complete graph $K_n$ on $n$ vertices.

A geometry $G$ is said to be upper homogeneous if for $k = 1, 2, \ldots, r(G)$, $G/x \cong G/y$ for every pairs $x, y$ of flats of rank $k$.

**Lemma 1.** If a geometry $G$ is upper homogeneous, has a modular copoint, and $|G| > r(G)$, then $G$ is non-splitting.

**Proof.** It suffices to show that if a geometry $G$ is upper homogeneous and has a modular copoint and is splitting, then $|G| = r(G)$.

Use induction on $n = r(G)$. For $n = 1, 2$, the lemma is true. Assume it holds for a geometry of rank less than $n$. Let $a$ be a point in $G$.

Suppose that $G/a$ is non-splitting. Since $G$ is splitting, we have $G = A \cup B$ where $A$ and $B$ are proper flats of $G$. Assume to the contrary that $A \cap B = \emptyset$. Let $x$ be a modular copoint of $G$. Then $x$ is splitting, i.e. $x = (A \cap x) \cup (B \cap x)$ where $A \cap x$ and $B \cap x$ are proper flats of $x$. Since $x$ is isomorphic to $G/a$ and $G/a$ is non-splitting, we have a contradiction. Thus $A \cap B \neq \emptyset$. Since $G/c = (A/c) \cup (B/c)$ for a point $c$ in $A \cap B$, it follows that $G/c$ is splitting. Since $G$ is upper homogeneous, we have a contradiction. Thus $G/a$ is splitting.

Note that $G/a$ is upper homogeneous and has a modular copoint. By the induction hypothesis, we have $|G/a| = r(G/a)$, i.e. $G/a$ is a Boolean algebra. Since $G/a$ is a Boolean algebra for every point $a$ in $G$, Theorem [2, p.89] and the scum theorem implies that $G$ is a Boolean algebra, i.e. $|G| = r(G)$.

**Lemma 2.** The following statements are equivalent.

1. $G$ is non-splitting
2. If $x$ is a copoint of $G$, then $G \setminus x$ contains a basis.
Proof. (1) \(\implies\) (2): Suppose that \(G \setminus x\) does not contain a basis. Then \(cl(G \setminus x)\) is a proper flat and \(G = x \cup cl(G \setminus x)\).

(2) \(\implies\) (1): Suppose that \(G\) is splitting. Let \(G = A \cup B\) where \(A\) and \(B\) are proper flats of \(G\). Let \(\alpha\) be a copoint of \(A\) containing \(A \cap B\) and let \(x = \alpha \cup B\) be a copoint of \(G\). Then \(G \setminus x\) is contained in \(A\). Since \(A\) is a proper flat, \(A\) does not contain a basis. Thus \(G \setminus x\) does not contain a basis.

Stonesifer and Bogart [7] proved the following theorem in terms of geometric lattices. Here we prove this by the previous results.

**Theorem 4.** If a geometry \(G\) has a modular copoint and \(L(G/a) \cong P_n\) for every point \(a\) in \(G\), then \(L(G) \cong P_{n+1}\) for \(n \geq 4\).

**Proof.** By Theorem 3, it suffices to show that \(|G| = \binom{n + 1}{2}\). Let \(x\) be a modular copoint in \(G\). Then the interval \([0, x]\) is isomorphic to \(L(G/a)\) for a point \(a\) not in \(x\). Thus \(|x| = |G/a| = |P_n| = \binom{n}{2}\). Since \(G\) contains no 4-point line as a minor by the scum theorem, \(G\) is binary. Also, since \(x\) is a modular copoint, no 2-point line is contained in \(G \setminus x\). If a line \(\ell\) is not in \(x\), then \(r(x \cap \ell) = r(x) + r(\ell) - r(x \cup \ell) = (n - 1) + 2 - n = 1\). It implies that every 3-point line (in \(G\)) not in \(x\) contains one point in \(x\).

Let \(|G \setminus x| = k\). Since \(G\) has a modular copoint and is upper homogeneous and \(|G| > |G/a| = \binom{n}{2}\) \(\geq n\) for \(n \geq 4\), by Lemma 1 and Lemma 2, we have \(k \geq n\). If \(|G| - |G/a| = k > n\), then Theorem 1 implies that \(G\) contains the Fano plane as a minor. But for \(n \geq 4\), \(G\) cannot contain the Fano plane as a minor by the scum theorem. Thus \(k = n\) and \(|G| = \binom{n + 1}{2}\).

Let \(p(G; \lambda)\) be the characteristic polynomial of a geometry \(G\). Then we have

\[
p(P_{n+1}; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n).
\]

In the next section, we characterize the partition lattice in terms of its characteristic polynomial and some additional conditions.
3. Main theorems

**Theorem 5.** If a geometry $G$ is upper homogeneous, has a modular copoint, and $p(G; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n)$, then $L(G) \cong P_{n+1}$.

*Proof.* Induction on $n$. For $n = 1, 2$, the theorem is true. Assume that it holds for a geometry of rank less than $n$.

Let $x$ be a modular copoint of $G$. By Lemma 1 and Lemma 2, we have $|G \setminus x| \geq n$. Since $G$ is modular, by the modular factorization theorem [5, Theorem 2], we have $|G \setminus x| \leq n$. Thus $|G \setminus x| = n$ and $p(x; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$. Since $[0, x]$ is isomorphic to $L(G/a)$ for some point $a$ not in $x$ and $G$ is upper homogeneous, we have $p(G/a; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$ for every point $a$ in $G$. Note that $G/a$ is upper homogeneous for every point $a$ in $G$. Since $x/b$ is a modular copoint of $G/b$ for a point $b$ in $x$, it follows that $G/a$ has a modular copoint for every point $a$ in $G$. By the induction hypothesis, $L(G/a) \cong P_n$ for every point $a$ in $G$. Thus Theorem 4 implies $L(G) \cong P_{n+1}$.

**Theorem 6.** If a geometry $G$ is non-splitting, supersolvable, and $p(G; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n)$, then $L(G) \cong P_{n+1}$.

*Proof.* We induct on the rank $n$ of a geometry $G$. For $n \leq 3$, the theorem is true. Assume it holds for a geometry of rank less than $n$.

Let $x$ be a modular copoint in a maximal chain of flats. Then we have $|G \setminus x| \leq n$. Since $G$ is non-splitting, Lemma 2 gives $|G \setminus x| \geq n$. Thus $|G \setminus x| = n$. By the modular factorization theorem, we have $p(x; \lambda) = (\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$. Suppose that $x$ is splitting. Let $x = A \cup B$ where $A$ and $B$ are proper flats of $x$. Since $L(G/a) \cong [0, x]$ for a point $a$ not in $x$, it follows that $G \cong \text{cl}(A \cup a) \cup \text{cl}(B \cup a)$ and so $G$ is splitting, a contradiction. Thus $x$ is non-splitting.

Now $x$ satisfies all the conditions of the theorem. By the induction hypothesis, we have $L(x) \cong P_n$. Note that $G \setminus x$ is exactly a basis of $G$. Since $x$ is a modular copoint with $|x| = \binom{n}{2}$, every two points of $G \setminus x$ is connected to a unique point in $x$ by a 3-point line. Therefore $L(G) \cong P_{n+1}$.
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References