ON OPERATORS WITH AN ABSOLUTE VALUE CONDITION

In Ho Jeon∗ and B. P. Duggal

Abstract. Let $\mathfrak{A}$ denote the class of bounded linear Hilbert space operators with the property that $|A^2| \geq |A|^2$. In this paper we show that $\mathfrak{A}$-operators are finitely ascensive and that, for non-zero operators $A$ and $B$, $A \otimes B$ is in $\mathfrak{A}$ if and only if $A$ and $B$ are in $\mathfrak{A}$. Also, it is shown that if $A$ is an operator such that $p(A)$ is in $\mathfrak{A}$ for a non-trivial polynomial $p$, then Weyl’s theorem holds for $f(A)$, where $f$ is a function analytic on an open neighborhood of the spectrum of $A$.

1. Introduction

Let $H$ be a Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on $H$. Recall ([1]) that an operator $A$ is $p$-hyponormal, $0 < p \leq 1$, if $|A^*|^{2p} \leq |A|^{2p}$. Evidently, 1-hyponormality is hyponormality. Let $H(p)$ denote the class of $p$-hyponormal operators. $H(\frac{1}{2})$ operators were first introduced by Xia (see [29]). The class of $H(p)$ operators, though strictly larger than the class of hyponormal operators ([5], [9], [29]), shares a large number of properties with hyponormal operators (see [1], [5], [7], [8]). We say that an operator $A \in \mathcal{B}(H)$ is paranormal if $A$ satisfies the norm condition $||A^2x|| \geq ||Ax||^2$ for all $x \in H$. An operator $A \in \mathcal{B}(H)$ is said to be normaloid if $||A|| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. It is well known that a $p$-hyponormal operator $A$ is paranormal and that a paranormal operator is normaloid.

Recently, Furuta-Ito-Yamazaki ([10]) have defined the following very interesting class of Hilbert space operators.

2000 Mathematics Subject Classification: 47B20.

Key words and phrases: class $A$ operator, polynomially class $A$ operator, tensor product, Weyl’s theorem.

∗ This work was supported by Korea Research Foundation Grant (KRF-2001-050-D0001).
Definition 1-1. The operator $A \in \mathcal{B}(H)$ is said to belong to Class A if $A$ satisfies an absolute value condition $|A^2| \geq |A|^2$.

In the following we denote “Class A” by simply $\mathfrak{A}$. In [10], it is shown that $\mathfrak{A}$ stands in the middle of classes of $p$-hyponormal and paranormal operators. More explicitly, we have the following inclusions:

$$\{\text{hyponormal operators}\} \subseteq \{\text{p-hyponormal operators}\} \subseteq \{\mathfrak{A} \text{-operators}\} \subseteq \{\text{paranormal operators}\} \subseteq \{\text{normaloid operators}\}.$$ 

It is well known that all of these inclusions may be proper (for details, see [9]). Ito ([16]) has shown that there are some parallelisms between absolute value conditions of $\mathfrak{A}$-operators and norm conditions of paranormal operators. Uchiyama ([26]) proved basic properties of $\mathfrak{A}$-operators and that Weyl’s theorem holds for $\mathfrak{A}$-operators.

Recall ([17], [18]) that the operator $A \in \mathcal{B}(H)$ is said to be finitely ascensive if for every $\lambda \in \mathbb{C}$ there is a $p \in \mathbb{N}$ such that

$$\ker(A - \lambda)^p = \ker(A - \lambda)^{p+1}.$$ 

The class of finitely ascensive operators is considerably large and plays a significant role in the study of local spectral theory (see [18], [20]). In section 2 we study basic properties of $\mathfrak{A}$-operators, which would make more explicit the relationship between the theory of $\mathfrak{A}$-operators and of paranormal operators. In particular, we prove that $\mathfrak{A}$-operators are finitely ascensive.

Given non-zero $A, B \in \mathcal{B}(H)$, let $A \otimes B$ denote the tensor product on the product space $H \otimes H$. The operation of taking tensor products $A \otimes B$ preserves many properties of $A, B \in \mathcal{B}(H)$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products, the spectraloid property is not (see [24, pp. 623 and 631]); again, whereas $A \otimes B$ is normal if and only if $A$ and $B$ are ([14], [25]), there exist paranormal operators $A$ and $B$ such that $A \otimes B$ is not paranormal ([2]). In section 3, for non-zero $A, B \in \mathcal{B}(H)$ it is shown that $A \otimes B \in \mathfrak{A}$ if and only if $A, B \in \mathfrak{A}$, which extends an analogous result on $p$-hyponormal operators in [7].

Recall ([12]) that an operator $A \in \mathcal{B}(H)$ is called Fredholm if it has closed range and finite dimensional null space and its range is of finite
co-dimension. The \textit{index} of a Fredholm operator $A \in \mathcal{B}(H)$ is given by
\[
\text{ind}(A) = \dim \ker(A) - \dim \ker(A^*).
\]
An operator $A \in \mathcal{B}(H)$ is called \textit{Weyl} if it is Fredholm of index zero. The \textit{Weyl spectrum} $\omega(A)$ of $A$ is defined by
\[
\omega(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl} \}.
\]
We write $\pi_0(A)$ for the set of eigenvalues of $A$ and $\pi_{00}(A)$ for the isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity. We say that \textit{Weyl’s theorem holds for} $A \in \mathcal{B}(H)$ if there is the equality
\[
\sigma(A) \setminus \omega(A) = \pi_{00}(A).
\]

\textbf{Definition 1-2.} An operator $A \in \mathcal{B}(H)$ is said to be a \textit{polynomially $A$-operator} if $p(A)$ is in $\mathfrak{A}$ with a non-trivial polynomial $p$.

In section 4, we show that Weyl’s theorem holds for $f(A)$ whenever $A$ is a polynomially $\mathfrak{A}$-operator and $f$ is an analytic function on an open neighborhood of $\sigma(A)$, which completely extends earlier results in [8] and [11] through slightly different approaches.

\section{Basic properties of $\mathfrak{A}$-operators}

First, we recall that a paranormal operator is normaloid ([15]), that a compact paranormal operator is normal ([15, Theorem 2] or [23]), and that scalar perturbations of paranormal operators are not paranormal as noted in [1, pp.174–175]. But as the case of hyponormal operator, if $A \in \mathcal{B}(H)$ is paranormal and $A - \lambda$ for any $\lambda \in \mathbb{C}$ is quasinilpotent, then $A = \lambda I$. Also, if $A \in \mathcal{B}(H)$ is paranormal, $\lambda \in \text{iso} \sigma(A)$ and $E_\lambda$ is the Riesz projection corresponding to $\lambda$, then $\text{ran}E_\lambda = \ker(A - \lambda)$ ([6] or [27]), which implies $A$ is isoloid (i.e., $\text{iso} \sigma(A) \subseteq \pi_0(A)$). Furthermore, if $\lambda \neq 0$ then $E_\lambda$ is self-adjoint and $\ker(A - \lambda) = \ker(A - \lambda)^*$ ([27]).

$\mathfrak{A}$-operators share these properties with paranormal operators and have the following result.

\textbf{Lemma 2-1.} ([26]) \textit{The following holds:}

(i) If $A \in \mathfrak{A}$, then the restriction $A|_M$ to its invariant subspace $M$ is also in $\mathfrak{A}$.

(ii) If $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $(A - \lambda)x = 0$ implies that $(A - \lambda)^*x = 0$. 

Remark 2-2. In the case of paranormal operators, the corresponding result of Lemma 2-1(i) follows immediately by its definition but that of Lemma 2-1(ii) does not. In fact, there is a counterexample given by A. Uchiyama ([28]). It looks very interesting and valuable.

The following result says that \( A \)-operators are finitely ascensive.

**Theorem 2-3.** Let \( A \in A \). Then

\[
\ker(A - \lambda) = \ker(A - \lambda)^2 \quad \text{for all} \quad \lambda \in \mathbb{C}.
\]

**Proof.** First, let \( \lambda = 0 \); if \( x \neq 0 \in \ker A^2 \), then we have

\[
0 = ||A^2 x|| ||x|| = ||A^2 x|| ||x||
\]

\[
\geq \langle |A^2 x, x| \rangle \geq \langle |A|^2 x, x \rangle
\]

\[
= ||A||^2 x ||x||^2.
\]

Second, let \( \lambda \neq 0 \in \mathbb{C} \); if \( x \neq 0 \in \ker (A - \lambda)^2 \), then by Lemma 2-1(ii) we have \((A - \lambda) x \in \ker (A - \lambda)^*\). Thus

\[
0 = ||(A - \lambda^*) x || ||x||
\]

\[
\geq \langle (A - \lambda^*) (A - \lambda) x, x \rangle
\]

\[
= ||(A - \lambda) x ||^2.
\]

Since (2.3.2) and (2.3.3) imply \( \ker (A - \lambda)^2 \subseteq \ker (A - \lambda) \) for all \( \lambda \in \mathbb{C} \) and \( \ker (A - \lambda) \subseteq \ker (A - \lambda)^2 \) in general, this completes the proof. \( \square \)

If \( A \in B(H) \) and \( F \) is a closed set in \( \mathbb{C} \), we define

\[
H_A(F) = \{ x \in H : \text{there exists an analytic } H\text{-valued function } f : \mathbb{C} \setminus F \rightarrow H \text{ such that } (A - \lambda) f(\lambda) = x \}.
\]

\( H_A(F) \) is said to be a spectral manifold of \( A \). If \( A \) has the single valued extension property, then the above definition is identical with \( H_A(F) = \{ x \in H : \sigma_A(x) \subseteq F \} \), where \( \sigma_A(x) \) is the local spectrum of \( A \) at \( x \) (see [20] for details).

**Corollary 2-4.** Let \( A \in A \) and \( \lambda \in \text{iso} \sigma(A) \). Then \( A \) has the single valued extension property and

\[
\text{ran} E_\lambda = \ker (A - \lambda) = H_A(\{\lambda\}),
\]

where \( E_\lambda \) is the Riesz projection corresponding to \( \lambda \).
Proof. Since $A$ is finitely ascensive, [18, Proposition 1.8] implies that $A$ has the single valued extension property. Combining [18, Corollary 2.4] and [27, Theorem 3.7] we easily have (2.3.4), and hence the proof is complete.

□

Remark 2-5. Proofs of Theorem 2-3 and Corollary 2-4 are thoroughly dependent on Lemma 2-1(ii). So we may notice it is impossible to get analogous results for paranormal operators. Actually, A. Uchiyama’s example ([28]) shows that (2.3.1) generally is not true for paranormal operators.

3. Tensor products of $\mathfrak{A}$-operators

In this section we completely extend earlier results on tensor products of $p$-paranormal operators in [7]. We start with

Lemma 3-1. ([25, Proposition 2.2]) Let $A_i, B_i \in \mathcal{B}(H)$ ($i = 1, 2$) be non-zero positive operators. Then the following conditions are equivalent:

(i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.
(ii) There exists $c > 0$ such that $A_1 \leq c A_2$ and $B_1 \leq c^{-1} B_2$.

Theorem 3-2. For non-zero $A, B \in \mathcal{B}(H)$ $A \otimes B \in \mathfrak{A}$ if and only if $A$ and $B \in \mathfrak{A}$.

Proof. Suppose $A \otimes B \in \mathfrak{A}$. Then

$$|A|^2 \otimes |B|^2 = |A \otimes B|^2 \leq |(A \otimes B)^2| = |A^2 \otimes B^2| = |A^2| \otimes |B^2|.$$  

Hence, by lemma 3.1, there exists a scalar $c > 0$ such that

$$|A|^2 \leq c |A^2|$$ and $$|B|^2 \leq c^{-1} |B^2|.$$  

This implies that

$$||A||^2 = \sup_{|x|=1} \langle |A|^2 x, x \rangle 
\leq \sup_{|x|=1} \langle c |A^2| x, x \rangle 
\leq c ||A^2|| = c ||A^2|| \leq c ||A||^2.$$
and

\[ ||B||^2 = \sup_{|x|=1} \langle B^2 x, x \rangle \leq \sup_{|x|=1} (c^{-1}|B^2|x, x) \leq c^{-1}||B^2|| = c^{-1}||B||^2 \leq c^{-1}||B||^2. \]

Clearly, we must have \( c = 1 \), and then \( A, B \in \mathfrak{A} \). Conversely, if \( A, B \in \mathfrak{A} \), then

\[(|A^2| - |B^2| \otimes (|B^2| - |B|^2)) \geq 0 \text{ implies } (|A^2| - |A|^2 \otimes |B|^2) \geq 0.\]

Hence \( A \otimes B \in \mathfrak{A} \). \( \square \)

For any \( X \in \mathcal{B}(H) \) let \( \tau_{AB^*} : \mathcal{B}(H) \to \mathcal{B}(H) \) be defined by \( \tau_{AB^*}(X) = AXB^* \) and \( C_2(H) \) denote the class of Hilbert-Schmidt operators on \( H \). Then we have

**Corollary 3-3.** For non-zero \( A, B \in \mathcal{B}(H) \), \( A, B \in \mathfrak{A} \) if and only if \( \tau_{AB^*}|_{C_2(H)} \in \mathfrak{A} \).

**Proof.** It is well known that the tensor product \( A \otimes B \) can be identified with the mapping \( \tau_{AB^*}|_{C_2(H)} \) (cf., [3, Lemma 2]). This completes the proof. \( \square \)

### 4. Polynomially \( \mathfrak{A} \)-operators

Let \( H(K) \) be the set of all analytic functions on an open neighborhood of compact subset \( K \subset \mathbb{C} \). In this section we prove that if \( A \) is a polynomially \( \mathfrak{A} \)-operator, then Weyl’s theorem holds for \( f(A) \) for \( f \in H(\sigma(A)) \). This extends well-known results of [8] and [11]: our proof however employs slightly different techniques.

**Theorem 4-1.** If \( A \in \mathcal{B}(H) \) is a polynomially \( \mathfrak{A} \)-operator and \( f \in H(\sigma(A)) \), then Weyl’s theorem holds for \( f(A) \).

The proof will be given by following several lemmas. We begin by elementary properties of polynomially \( \mathfrak{A} \)-operators.
Lemma 4-2. Let $A$ be a polynomially $\mathfrak{A}$-operator. Then the following holds.

(i) If $A$ is quasinilpotent, then $A$ is nilpotent.
(ii) $A$ is isoloid.
(iii) $A$ is finitely ascensive.

Proof. Towards (i), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial $p$. We may write

$$p(\lambda) - p(0) = a_0 \lambda^m \prod_{i=1}^{n} (\lambda - \lambda_i)$$

for some scalars $a_0, \lambda_1, \ldots, \lambda_n$ and integers $m, n$. If $A$ is quasinilpotent, then

$$\sigma(p(A)) = p(\sigma(A)) = p(0),$$

so that $p(A) - p(0)$ is also quasinilpotent. Thus it follows that

$$p(A) - p(0) = a_0 A^m \prod_{i=1}^{n} (A - \lambda_i) = 0.$$  

Since $A - \lambda_i$ is invertible for every $1 \leq i \leq n$, we have that $A^m = 0$.

Towards (ii), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial $p$. Let $\lambda \in \text{iso}(A)$. Then using the spectral decomposition, we can represent $A$ as the direct sum $A = A_1 \oplus A_2$, where $\sigma(A_1) = \{\lambda\}$ and $\sigma(A_2) = \sigma(A) \setminus \{\lambda\}$. Since $p(A_1)$ is also $\mathfrak{A}$-operator by Lemma 2-1(i), the quasinilpotency of $p(A_1) - p(\lambda)$ implies the nilpotency of $A_1 - \lambda$ from similar arguments of proof of (i). Therefore $\lambda \in \pi_0(A_1)$ and hence $\lambda \in \pi_0(A)$. This shows that $A$ is isoloid.

Towards (iii), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial $p$. If $\lambda \in \sigma(A)$, then we may assume that for some scalars $a_0, \lambda_1, \ldots, \lambda_n$ and integers $m, n$

$$p(A) - p(\lambda) = a_0 (A - \lambda)^m \prod_{i=1}^{n} (A - \lambda_i).$$

Let $x(\neq 0) \in \ker(A - \lambda)^{m+1}$. Then

$$(p(A) - p(\lambda))x = b(A - \lambda)^m x$$

for some scalar $b$. 


Let $p(\lambda) = 0$;

$$
0 = \|(A - \lambda)^{2m}x\||\|x\|
= ||b^{-2}p(A)^{2}|x||\|x||
\geq \langle b^{-2}p(A)^{2}|x, x\rangle
\geq \langle ||b^{-1}p(A)^{2}|x, x\rangle
= ||b^{-1}p(A)x||^2
= \|(A - \lambda)^{m}x\||^2.
$$

(4.2.3)

Let $p(\lambda) \neq 0$; since by Lemma 2-1(ii)

$$(p(A) - p(\lambda))(A - \lambda)^{m}x = 0 \Rightarrow (p(A) - p(\lambda))^*(A - \lambda)^{m}x = 0,$$

we have

$$
\|(A - \lambda)^{m}x\||^2 = \langle (A - \lambda)^{m}x, (A - \lambda)^{m}x\rangle
= \langle b^{-1}(p(A) - p(\lambda))x, (A - \lambda)^{m}x\rangle
\langle x, b^{-1}(p(A) - p(\lambda))^*(A - \lambda)^{m}x\rangle
= 0.
$$

(4.2.4)

Thus (4.2.3) and (4.2.4) implies that $x \in \ker(A - \lambda)^{m}$. Therefore

$$
\ker(A - \lambda)^{m+1} \subseteq \ker(A - \lambda)^{m}
$$

and the reverse inclusion is evident. This completes the proof. 

In view of Remark 2-5, it also seems to be impossible to get Lemma 4-2(iii) in the context of (polynomially) paranormal operators.

**Lemma 4-3.** ([17, Theorem 2]) Let $A \in \mathcal{B}(H)$ be finitely ascensive. Then Weyl’s theorem holds for $A$ if and only if $\text{ran}(A - \lambda)$ is closed for every $\lambda \in \pi_{00}(A)$.

**Proposition 4-4.** Weyl’s theorem holds for every polynomially $\mathfrak{A}$-operators.

**Proof.** Let $A$ be a polynomially $\mathfrak{A}$-operator. Then by Lemma 4-2(iii) $A$ is finitely ascensive. So it suffices to show that $\text{ran}(A - \lambda)$ is closed for every $\lambda \in \pi_{00}(A)$ by Lemma 4-3. Suppose $\lambda \in \pi_{00}(A)$ and let $E_{\lambda}$ be the Riesz projection with corresponding to $\lambda$. Then $\text{ran}(E_{\lambda})$ is finite.
dimensional because \((A - \lambda)_{\text{ran}E_\lambda}\) is nilpotent as shown in the proof of Lemma 4-2(ii), and
\[
0 < \dim \ker(A - \lambda)_{\text{ran}E_\lambda} = \dim \ker(A - \lambda) < \infty.
\]
From [18, Corollary 2.4] we have \(\text{ran}E_\lambda = H_A(\{\lambda\})\), and so [19, Lemma 2] implies that \(A - \lambda\) is Fredholm. Hence \(\text{ran}(A - \lambda)\) is closed for \(\lambda \in \pi_{00}(A)\).

We show that the Weyl spectrum obeys the spectral mapping theorem for polynomially \(\mathfrak{A}\)-operators.

**Lemma 4-5.** If \(A \in \mathcal{B}(H)\) is a polynomially \(\mathfrak{A}\)-operator, then
\[
\omega(f(A)) = f(\omega(A)) \quad \text{for every } f \in H(\sigma(A)).
\]

**Proof.** First, let \(f\) be a polynomial. Since it is well known ([4, Theorem 3.2]) that \(\omega(f(A)) \subseteq f(\omega(A))\), in view of [13, Theorem 5], it suffices to show that
\[
\text{ind}(A - \lambda I) \text{ ind}(A - \mu I) \geq 0 \quad \text{for each pair } \lambda, \mu \in \mathbb{C} \setminus \sigma_e(A).
\]
By Lemma 4-2(iii), \(A - \lambda I\) has finite ascent for every \(\lambda \in \mathbb{C}\). Observe that if \(A - \lambda\) is Fredholm of finite ascent, then \(\text{ind}(A - \lambda) \leq 0\) by the same arguments in the proof of [13, Theorem 3]. Thus we can see that (4.5.2) holds for every polynomially \(\mathfrak{A}\)-operators \(T\). This proves that the equality (4.5.1) holds for every polynomial \(f\). Now the equality (4.5.1) for \(f \in H(\sigma(A))\) follows at once from an argument of Oberai ([22, Theorem 2]).

Now, we conclude this paper with the proof of Theorem 4-1.

**Proof of Theorem 4-1.** Remembering [21, Lemma] that if \(A\) is isoloid, then
\[
f(\sigma(A) \setminus \pi_{00}(A)) = \sigma(f(A) \setminus \pi_{00}(f(A)) \quad \text{for every } f \in H(\sigma(A));
\]
it follows from Lemma 4-2(ii), Proposition 4-4 and Lemma 4-5 that
\[
\sigma(f(A)) \setminus \pi_{00}(f(A)) = f(\sigma(A) \setminus \pi_{00}(A)) = f(\omega(A)) = \omega(f(A)),
\]
which implies that Weyl’s theorem holds for \(f(A)\).

**Acknowledgement.** The authors would like to express their cordial thanks to the referee for his kind suggestions and also to Professor A. Uchiyama for sending his valuable preprints on paranormal operators.
References

On operators with an absolute value condition


In Ho Jeon
Department of Mathematics
Ewha Women’s University
Seoul 120-750, Korea
E-mail: jih@math.ewha.ac.kr

B. P. Duggal
Department of Mathematics
UAEU
P. O. Box 17551, Al Ain
Arab Emirates
E-mail: bpduggal@uaeu.ac.ae