A NOTE ON NULL DESIGNS
OF DUAL POLAR SPACES

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Abstract. Null designs on the poset of dual polar spaces are considered. A poset of dual polar spaces is the set of isotropic subspaces of a finite vector space equipped with a nondegenerate bilinear form, ordered by inclusion. We show that the minimum number of isotropic subspaces to construct a nonzero null $t$-design is $\prod_{i=0}^{t}(1 + q^i)$ for the types $B_N$, $D_N$, whereas for the case of type $C_N$, more isotropic subspaces are needed.

1. Introduction

Null designs are defined on ranked partially ordered sets. Let $P$ be a finite ranked partially ordered set. Given two ranks of $P$, $t \leq k$, we can form a 0,1 matrix, called an adjacency matrix, with columns indexed by the elements of rank $k$ and the rows indexed by the elements of rank $t$. The kernel of this adjacency matrix forms a space of very interesting objects called null $(t, k)$-designs [8]. The poset of dual polar spaces of a given type is the set of isotropic subspaces of a finite vector space equipped with the nondegenerate bilinear form of the corresponding type, ordered by inclusion. In this paper, we consider the space of null $(t, k)$-designs of posets of dual polar spaces of type $B_N$, $C_N$ and $D_N$ [2], [12], [13]. Note that the corresponding Chevalley group has a natural action on the space of null $(t, k)$-designs, that is, the space of null $(t, k)$-designs forms a representation of the corresponding Chevalley group. These representations are considered in [12].

We are especially interested in the number of nonzero entries, called the support size, of nonzero null $t$-designs. P. Frankl and J. Pach [7]
proved that the minimum support size of non-zero null $t$-designs is $2^{t+1}$ for the Boolean algebras, and the minimal null designs are characterized for some special cases [4], [10]. It is proved that $\prod_{i=0}^{t}(1 + q^{i})$ is the minimum support size of non-zero null $t$-designs of the lattices of subspaces of a finite vector space in [6], when $k = t + 1$, and the minimal null designs are characterized in [5]. Moreover, there is a general theorem which gives a lower bound for the minimum support size of non-zero null $t$-designs [4]. In this article, we apply known theorems on the support size of non-zero null $t$-designs to the poset of dual polar spaces of type $B_N$, $C_N$ and $D_N$. In Section 2, we state some known theorems on the number of elements needed to construct a nonzero null design. In Section 3, we apply the results in Section 2 to the posets of dual polar spaces of type $B_N$, $C_N$ and $D_N$.

2. Preliminaries

In this section, we give basic definitions and state some known theorems.

For a finite set $X$, we let $\mathbb{R}[X] = \{ \sum_{x \in X} c_x x : c_x \in \mathbb{R} \}$ denote the vector space over the real field $\mathbb{R}$ with a basis $X$. If $P$ is a finite ranked partially ordered set, then we let $X_i^P$ be the set of elements of rank $i$ of $P$ and define the linear map $d_{i,j}^P : \mathbb{R}[X_i^P] \to \mathbb{R}[X_j^P]$, $j \leq i$, as follows;

$$d_{i,j}^P(x) = \sum_{y \leq x \in X_j^P} y.$$ 

For integers $0 \leq t < k$, null $(t, k)$-design of a finite ranked poset $P$ is an element of the kernel of $d_{k,t}^P$. We will use $N_P(t, k)$ for the vector space of null $(t, k)$-designs of $P$.

For a finite ranked poset $P$, we say that $P$ satisfies the downmap condition, if the following condition is satisfied;

$$\text{for all } t < k, \quad d_{k,t}^P(x) = 0 \quad \text{implies} \quad d_{k,t'}^P(x) = 0 \quad \text{if } t' \leq t.$$ 

For an element $\omega \in \mathbb{R}[X]$ and $x \in X$, $c_\omega(x)$ is the coefficient of $x$ in $\omega$ and the support of $\omega$ is the set $\text{Supp}(\omega) = \{ x \in X | c_\omega(x) \neq 0 \}$. A minimal null design is a nonzero null design with the minimum support size. In the following propositions, we suppose that $P$ is a finite ranked meet semilattice with the downmap condition, where $\mu_P$ denote the Möbius function defined on $P$, and for $x, y \in P$, $x \wedge y$ is the meet of $x$ and $y$. We refer to [11] for the definitions of the meet semilattice and the Möbius
function. The following propositions provide general rules to find the minimum support size of nonzero null \( t \)-designs and to characterise the minimum null \( t \)-designs, whose proofs are in [6].

**Proposition 1.** If \( \omega \in N_P(t, t+1) \), \( \omega \neq 0 \), then
\[
|\text{Supp}(\omega)| \geq \min_{y \in X_{t+1}^P} \left( \sum_{z \leq y} |\mu_P(z, y)| \right).
\]

**Proposition 2.** If the lower bound in Proposition 1 gives the tight bound, then the coefficients of a nonzero minimal null design in \( N_P(t, t+1) \) are \( \pm c \) or 0 for some nonzero constant \( c \in \mathbb{R} \). Moreover, if \( \omega \in N_P(t, t+1) \) is a minimal null design, then for each \( y \in \text{Supp}(\omega) \) and \( z \leq y \), there must be exactly \( |\mu_P(z,y)| \) many \( x \in \text{Supp}(\omega) \) such that \( x \land y = z \) and \( c_\omega(x) = \text{sign}(\mu_P(z,y))c_\omega(y) \).

3. Null designs of dual polar spaces

In this section, we apply the propositions stated in Section 2 to the lattice of subspaces of a finite vector space and to posets of dual polar spaces. We first define posets of isotropic spaces with respect to nondegenerate bilinear form of types \( B_N, C_N, \) and \( D_N \) (see [1],[3]). Let \( q \) be a power of a prime number, and each vector space has the Galois field \( \mathbb{F}_q \) as its base field.

**Definition 3.** (1) A \((2N + 1)\)-dimensional vector space \( V_N \) over \( \mathbb{F}_q \) is of type \( B_N \), if it has a basis \( \{e_1, \ldots, e_N, e_{-1}, \ldots, e_{-N}, e_0\} \) with a symmetric bilinear form \( B \), whose Gram matrix is
\[
\begin{bmatrix}
0 & I_N & 0 \\
I_N & 0 & \\
0 & \ldots & 1
\end{bmatrix}.
\]

(2) A \(2N\)-dimensional vector space \( V_N \) over \( \mathbb{F}_q \) is of type \( C_N \), if it has a basis \( \{e_1, \ldots, e_N, e_{-1}, \ldots, e_{-N}\} \) with a skew symmetric bilinear form \( B \), whose Gram matrix is
\[
\begin{bmatrix}
0 & \ I_N \\
-\ I_N & 0
\end{bmatrix}.
\]

(3) A \(2N\)-dimensional vector space \( V_N \) over \( \mathbb{F}_q \) is of type \( D_N \), if it has a basis \( \{e_1, \ldots, e_N, e_{-1}, \ldots, e_{-N}\} \) with a symmetric bilinear form \( B \),
whose Gram matrix is 
\[
\begin{bmatrix}
0 & I_N \\
I_N & 0
\end{bmatrix}.
\]

**Definition 4.** For each type of vector spaces $V_N$ defined in Definition 3, define a poset as the set of isotropic subspaces of $V_N$ with respect to the given bilinear form $B$, ordered by inclusion. Let us call these posets $P_{B_N}$, $P_{C_N}$ and $P_{D_N}$ depending on the type of given vector space $V_N$.

Note that, by Witt’s theorem, each poset in Definition 4 has maximal rank $N$ and that a subspace of an isotropic subspace of $V_N$ is again an isotropic subspace of $V_N$. Therefore, $P_{B_N}$, $P_{C_N}$ and $P_{D_N}$ are meet semilattices. We let $L_n(q)$ denote the lattice of subspaces of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$. Note that $\mu_P(z, y) = (−1)^{i−j}q^{(i_0^2)}$ when $P = L_n(q)$ and $z \in P_j$, $y \in P_i$ (see [11]). It is known that the signed sum of maximal isotropic subspaces of $V_N$ of type $D_N$ forms a null $(N−1, N)$-design, where the sign of each isotropic space is defined as $\text{sign}(y) = (−1)^{\dim(y \wedge y_0)}$ for some fixed isotropic space $y_0$ (see [9]). If we apply Propositions 1 and 2 to $L_n(q)$, we obtain the following result that serves as the fundamental case for the posets of dual polar spaces. The proofs are in [5] and [6].

**Proposition 5.** If $P$ is the subspace lattice of an $n$-dimensional vector space $V$, i.e. $P = L_n(q)$,

1. the minimum of the support size of non-zero elements of $N_P(t, t+1)$ is $\prod_{i=0}^t (1 + q^i)$,

2. if $\omega$ is a minimal null design in $N_P(t, t+1)$, then $\omega$ is a multiple of the signed sum of maximal isotropic subspaces of some $2(t+1)$-dimensional subspace of $V$ equipped with the symmetric bilinear form of type $D_{t+1}$.

Observe that for a given isotropic subspace $x$ of $V_N$ of types $B_N$, $C_N$ and $D_N$, the interval $[\langle 0 \rangle, x]$ is exactly same as the interval $[(0), x]$ in $L_n(q)$, where $n = 2N + 1$ or $n = 2N$ depending on the type. Hence, the value of Möbius functions on $P_{B_N}$, $P_{C_N}$ and $P_{D_N}$ equals to the value of Möbius function on $L_n(q)$, and the lower bounds given in Proposition 1 are $\prod_{i=0}^t (1 + q^i)$. We now show that this bound is tight for $P_{B_N}$, $P_{D_N}$, but it is not tight for $P_{C_N}$.

**Theorem 6.** If $P = P_{B_N}, P_{D_N}$, the minimum of the support size of non-zero elements of $N_P(t, t+1)$ is $\prod_{i=0}^t (1 + q^i)$. 

Throughout the proof, we let \( x \) get a contradiction. Without loss of generality, we may assume that \( \prod_{i=1}^t \langle i \rangle \) for some cases of type \( B \). Let \( \langle i \rangle \) for \( t \) be elements in \( N(q) \). We let \( x \cup y \) be the join of \( x \) and \( y \), that is the smallest space that contains \( x \) and \( y \). The meet of \( x \) and \( y \), denoted by \( x \land y \), is the intersection of \( x \) and \( y \).

**Theorem 7.** Let \( t + 1 = N \), \( N > 1 \), then for \( \omega \in N_{P_{CN}}(t, t+1) \), which is non-zero,

\[
|\text{Supp}(\omega)| > \prod_{i=0}^t (1 + q^i).
\]

**Proof.** Throughout the proof, we let \( P = P_{CN} \). Let us assume that there is a non-zero element \( \omega \) of \( N_P(t, t+1) \), whose support size \( \prod_{i=0}^t (1 + q^i) \). We apply Proposition 2 to \( \omega \) throughout the proof to get a contradiction. Without loss of generality, we may assume that \( x_0 = \langle e_1, \ldots, e_{t+1} \rangle \in P_{t+1} \) is in \( \text{Supp}(\omega) \) and \( c_\omega(x_0) = +1 \). Let \( z_0 = \langle e_1, \ldots, e_{t-1} \rangle \in P_{t-1} \) and \( y_1 = z_0 \cup \langle e_t \rangle \). Let \( y_2 = z_0 \cup \langle e_{t+1} \rangle \) and \( y_3 = z_0 \cup \langle e_t, e_{t+1} \rangle \) be elements in \( P_t \). Then, since \( y_i \leq x_0 \) for each \( i = 1, 2, 3 \), by Proposition 2, there must be unique \( x_i \) in \( \text{Supp}(\omega) \) such that \( x_0 \land x_i = y_i \) and \( c_\omega(x_i) = -1 \). We also let \( A = \{ x \in \text{Supp}(\omega) \mid x_0 \land x = z_0 \} \), then by Proposition 2, \( |A| = q \) and \( c_\omega(x) = +1 \) for all \( x \in A \).

Choose two vectors \( w_1, w_2 \) so that \( x_i = y_i \cup \langle w_i \rangle \) for \( i = 1, 2 \), and \( x_i^A \in A \), then \( x_0 \land x_i^A \), \( i = 1, 2 \), is \( t \)-dimensional since both \( x_i \) and \( x_i^A \) contain \( z_0 \) in \( P_{t-1} \), and \( c_\omega(x_i) = -c_\omega(x_i^A) \). Hence \( x_1^A \land x_1 = z_0 \cup \langle v_1 \rangle \) for some \( v_1 = w_1 + \alpha e_t \), and \( x_2^A \land x_2 = z_0 \cup \langle v_2 \rangle \) for some \( v_2 = w_2 + \beta e_{t+1} \), \( \alpha, \beta \in F_q \) (note that \( x_1^A \land x_1 \) can not be \( y_i \) for \( i = 1, 2 \)). Observe that \( x_i = y_i \cup \langle w_i \rangle = y_i \cup \langle v_i \rangle \) for \( i = 1, 2 \), and \( x_1 \land x_1^A = z_0 \cup \langle v_1, v_2 \rangle \).

Since \( t + 1 = N \), \( x_1 \) and \( x_2 \) are maximal isotropic spaces and \( x_1 \cup \langle v_2 \rangle \) and \( x_2 \cup \langle v_1 \rangle \) are not isotropic spaces. Therefore, \( B(e_t, v_2) \) and \( B(e_{t+1}, v_1) \) are non-zero and without loss of generality, we assume that \( B(e_t, v_2) = B(e_{t+1}, v_1) = 1 \). Let us consider another element \( x_2^A \) of \( A \),
then by the same argument as above, $x_2^A = z_0 \lor \langle v'_1, v'_2 \rangle$ where $v'_1 = v_1 + \alpha' e_t$ and $v'_2 = v_2 + \beta' e_{t+1}$, for $\alpha', \beta' \in \mathbb{F}_q$. Since $x_2^A$ is isotropic, $B(v'_1, v'_2) = B(v_1 + \alpha' e_t, v_2 + \beta' e_{t+1}) = \beta'B(v_1, e_{t+1}) + \alpha'B(e_t, v_2) = -\beta' + \alpha'$ should be 0. Hence, we have $\alpha' = \beta'$. Above observation implies that $A = \{z_0 \lor (v_1 + \gamma e_t, v_2 + \gamma e_{t+1}) | \gamma \in \mathbb{F}_q\}$, since $|A| = q$.

We now consider $x_3 \in \text{Supp}(\omega)$ and $x_3^A = z_0 \lor \langle v_1 + e_t, v_2 + e_{t+1} \rangle \in A$. Since $x_3 \land x_1^A$ is $t$-dimensional, $x_3 = y_3 \lor \langle v_3 \rangle$, where $v_3 = \alpha_1 v_1 + \alpha_2 v_2$, $\alpha_1, \alpha_2 \in \mathbb{F}_q$. $x_3$ is an isotropic space, so we have $B(e_t + e_{t+1}, v_3) = \alpha_2 + \alpha_1 = 0$ and, we can say that $x_3 = z_0 \lor \langle e_t + e_{t+1}, v_1 - v_2 \rangle$. Moreover, since $x_3 \land x_3^A$ is $t$-dimensional, $\langle v_1 + e_t, v_2 + e_{t+1} \rangle \land \langle e_t + e_{t+1}, v_1 - v_2 \rangle$ should be 1-dimensional but it is $\langle 0 \rangle$, so we have a contradiction. This completes the proof.

**Remark 8.** Note that the skew symmetry of the bilinear form of type $C_N$ plays the central role in the proof of Theorem 7.

**References**


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