ON $(\alpha, \beta)$-FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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ABSTRACT. Using the belongs to relation ($\in$) and quasi-coincidence with relation (q) between fuzzy points and fuzzy sets, the concept of $(\alpha, \beta)$-fuzzy subalgebras where $\alpha, \beta$ are any two of \{\in, q, \in \lor q, \in \land q\} with $\alpha \neq \in \land q$ is introduced, and related properties are investigated.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [4], played a vital role to generate some different types of fuzzy subgroups, called $(\alpha, \beta)$-fuzzy subgroups, introduced by Bhakat and Das[2]. In particular, $(\in, \in \lor q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy sub-systems of other algebraic structures. With this objective in view, we introduce the concept of $(\alpha, \beta)$-fuzzy subalgebra of a $BCK/BCI$-algebra and investigate related results.

2. Preliminaries

By a $BCI$-algebra we mean an algebra $(X, \ast, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X) \left( ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \right)$,
(ii) $(\forall x, y \in X) \left( (x \ast (x \ast y)) \ast y = 0 \right)$,
(iii) $(\forall x \in X) \left( x \ast x = 0 \right)$,
(iv) $(\forall x, y \in X) \left( x \ast y = y \ast x = 0 \Rightarrow x = y \right)$.

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We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x \ast y = 0$. If a $BCI$-algebra $X$ satisfies $0 \ast x = 0$ for all $x \in X$, then we say that $X$ is a $BCK$-algebra. In what follows let $X$ denote a $BCK/BCI$-algebra unless otherwise specified. A nonempty subset $S$ of $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$. We refer the reader to the book [3] for further information regarding $BCK/BCI$-algebras.

A fuzzy set $\mu$ in a set $X$ of the form

$$
\mu(y) := \begin{cases} 
  t & \text{if } y = x, \\
  0 & \text{if } y \neq x
\end{cases}
$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu[4] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \lor q, \in \land q\}$.

To say that $x_t \in \mu \lor q \mu$ (resp. $x_t \in \land q \mu$) means that $x_t \in \mu$ or $x_t \in \mu \land q \mu$. For all $t_1, t_2 \in [0, 1]$, $\min\{t_1, t_2\}$ will be denoted by $M(t_1, t_2)$.

A fuzzy set $\mu$ in $X$ is called a fuzzy subalgebra of $X$ if it satisfies

$$
(\forall x, y \in X) \ (\mu(x \ast y) \geq M(\mu(x), \mu(y))).
$$

**Proposition 2.1.** Let $\mu$ be a fuzzy set in $X$. Then $\mu$ is a fuzzy subalgebra of $X$ if and only if $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$ is a subalgebra of $X$ for all $t \in [0, 1]$, for our convenience, the empty set $\emptyset$ is regarded as a subalgebra of $X$.

**3. ($\alpha, \beta$)-fuzzy subalgebras**

In what follows let $\alpha$ and $\beta$ denote any one of $\in, q, \in \lor q, \in \land q$ unless otherwise specified. To say that $x_t \alpha \mu$ means that $x_t \alpha \mu$ does not hold.

**Proposition 3.1.** For any fuzzy set $\mu$ in $X$, the condition (1) is equivalent to the following condition

$$
(\forall x, y \in X) \ (\forall t_1, t_2 \in (0, 1)) \ (x_{t_1}, y_{t_2} \in \mu \Rightarrow (x \ast y)_{M(t_1, t_2)} \in \mu).
$$

**Proof.** Assume that the condition (1) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0, 1)$ be such that $x_{t_1}, y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Then...
which imply from (1) that
\[ \mu(x + y) \geq M(\mu(x), \mu(y)) \geq M(t_1, t_2). \]

Hence \((x + y)_{M(t_1,t_2)} \in \mu\).

Conversely suppose that the condition (2) is valid. Note that \(x_{\mu(x)} \in \mu\) and \(y_{\mu(y)} \in \mu\) for all \(x, y \in X\). Thus \((x + y)_{M(\mu(x),\mu(y))} \in \mu\) by (2), and so \(\mu(\mu(x), \mu(y))\).

Note that if \(\mu\) is a fuzzy set in \(X\) defined by \(\mu(x) \leq 0.5\) for all \(x \in X\), then the set \(\{x \in X \mid x \in \mu\}\) is empty.

A fuzzy set \(\mu\) in \(X\) is said to be an \((\alpha, \beta)\)-fuzzy subalgebra of \(X\), where \(\alpha \neq 0\) if it satisfies the following conditions:

\[ (3) \quad (\forall x, y \in X) ((\forall t_1, t_2 \in (0,1))(x_{t_1, \alpha \mu}, y_{t_2, \alpha \mu} \Rightarrow (x + y)_{M(t_1,t_2)}(\beta \mu). \]

**Example 3.2.** Consider a BCI-algebra \(X = \{0, a, b, c\}\) with the following Cayley table (see [1]):

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
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<tr>
<td>a</td>
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<td>0</td>
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<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \(\mu\) be a fuzzy set in \(X\) defined by \(\mu(0) = 0.6\), \(\mu(a) = 0.7\), and \(\mu(b) = \mu(c) = 0.3\). Then \(\mu\) is an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\). But

1. \(\mu\) is not an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\) since \(a_{0.62} \in \mu\) and \(a_{0.66} \in \mu\), but \((a * a)_{M(0.62,0.66)} = a_{0.62} \notin \mu\).
2. \(\mu\) is not a \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\) since \(a_{0.41} q \mu\) and \(b_{0.77} q \mu\), but \((a * b)_{M(0.41,0.77)} = c_{0.41} \notin \varepsilon q \mu\).
3. \(\mu\) is not an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\) since \(a_{0.5} \in \varepsilon q \mu\) and \(|0.8 \in \varepsilon q \mu|\), but \((a * c)_{M(0.5,0.8)} = b_{0.5} \notin \varepsilon q \mu\).

**Theorem 3.3.** Every \((\varepsilon, \varepsilon)\)-fuzzy subalgebra is an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra.

**Proof.** Let \(\mu\) be an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\). Let \(x, y \in X\) and \(t_1, t_2 \in (0,1]\) be such that \(x_{t_1, \mu} \in \mu\) and \(y_{t_2, \mu} \in \mu\). Then \(x_{t_1, \varepsilon q} \mu\) and \(y_{t_2, \varepsilon q} \mu\), which imply that \((x + y)_{M(t_1,t_2)} \in \varepsilon q \mu\). Hence \(\mu\) is an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra of \(X\).

**Theorem 3.4.** Every \((\varepsilon, \varepsilon)\)-fuzzy subalgebra is an \((\varepsilon, \varepsilon)\)-fuzzy subalgebra.

**Proof.** Straightforward.
Example 3.2 shows that the converse of Theorems 3.3 and 3.4 need not be true.

**Proposition 3.5.** If $\mu$ is a non-zero $(\alpha, \beta)$-fuzzy subalgebra of $X$, then $\mu(0) > 0$.

**Proof.** Assume that $\mu(0) = 0$. Since $\mu$ is non-zero, there exists $x \in X$ such that $\mu(x) = t > 0$. If $\alpha = \varepsilon$ or $\alpha = \varepsilon \lor q$, then $x_1 \alpha \mu$, but $(x \ast x)_{M(t,1)} = 0_1 \beta \mu$. This is a contradiction. If $\alpha = q$, then $x_1 \alpha \mu$ because $\mu(x) + 1 = t + 1 > 1$. But $(x \ast x)_{M(1,1)} = 0_1 \beta \mu$, which is a contradiction. Hence $\mu(0) > 0$.

For a fuzzy set $\mu$ in $X$, we denote $X_0 := \{x \in X \mid \mu(x) > 0\}$.

**Theorem 3.6.** If $\mu$ is a nonzero $(\varepsilon, \varepsilon)$-fuzzy subalgebra of $X$, then the set $X_0$ is a subalgebra of $X$.

**Proof.** Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Suppose that $\mu(x \ast y) = 0$. Note that $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$, but $(x \ast y)_{M(\mu(x),\mu(y))} \subseteq \mu$ because $\mu(x \ast y) = 0 < M(\mu(x),\mu(y))$. This is a contradiction, and thus $\mu(x \ast y) > 0$, which shows that $x \ast y \in X_0$. Consequently $X_0$ is a subalgebra of $X$.

**Theorem 3.7.** If $\mu$ is a nonzero $(\varepsilon, q)$-fuzzy subalgebra of $X$, then the set $X_0$ is a subalgebra of $X$.

**Proof.** Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. If $\mu(x \ast y) = 0$, then

$$\mu(x \ast y) + M(\mu(x),\mu(y)) = M(\mu(x),\mu(y)) \leq 1.$$ 

Hence $(x \ast y)_{M(\mu(x),\mu(y))} \subseteq \mu$, which is a contradiction since $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$. Thus $\mu(x \ast y) > 0$, and so $x \ast y \in X_0$. Therefore $X_0$ is a subalgebra of $X$.

**Theorem 3.8.** If $\mu$ is a nonzero $(q, \varepsilon)$-fuzzy subalgebra of $X$, then the set $X_0$ is a subalgebra of $X$.

**Proof.** Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, which imply that $x_1 q \mu$ and $y_1 q \mu$. If $\mu(x \ast y) = 0$, then $\mu(x \ast y) < 1 = M(1,1)$. Therefore $(x \ast y)_{M(1,1)} \subseteq \mu$, which is a contradiction. It follows that $\mu(x \ast y) > 0$ so that $x \ast y \in X_0$. This completes the proof.

**Theorem 3.9.** If $\mu$ is a nonzero $(q, q)$-fuzzy subalgebra of $X$, then the set $X_0$ is a subalgebra of $X$. 

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, and therefore $x_1 q \mu$ and $y_1 q \mu$. If $\mu(x * y) = 0$, then $\mu(x * y) + M(1, 1) = 0 + 1 = 1$, and so $(x * y)_{M(1,1)} \notq \mu$. This is impossible, and hence $\mu(x * y) > 0$, i.e., $x * y \in X_0$. This completes the proof.

**Corollary 3.10.** If $\mu$ is one of the following

(i) a nonzero $(\in, \in \land q)$-fuzzy subalgebra of $X$,
(ii) a nonzero $(\in, \in \lor q)$-fuzzy subalgebra of $X$,
(iii) a nonzero $(\in \lor q, q)$-fuzzy subalgebra of $X$,
(iv) a nonzero $(\in \lor q, \in q)$-fuzzy subalgebra of $X$,
(v) a nonzero $(\in q, \in \lor q)$-fuzzy subalgebra of $X$,
(vi) a nonzero $(q, \in \lor q)$-fuzzy subalgebra of $X$,
(vii) a nonzero $(q, \in \land q)$-fuzzy subalgebra of $X$,
then the set $X_0$ is a subalgebra of $X$.

**Proof.** The proof is similar to the proof of Theorems 3.6, 3.7, 3.8, and/or 3.9.

**Theorem 3.11.** Every nonzero $(q, q)$-fuzzy subalgebra of $X$ is constant on $X_0$.

**Proof.** Let $\mu$ be a nonzero $(q, q)$-fuzzy subalgebra of $X$. Assume that $\mu$ is not constant on $X_0$. Then there exists $y \in X_0$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Then either $t_y > t_0$ or $t_y < t_0$. Suppose $t_y < t_0$ and choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_y < t_2$. Then $\mu(0) + t_1 = t_0 + t_1 > 1$ and $\mu(y) + t_2 = t_y + t_2 > 1$, and so $t_1 q \mu$ and $y_1 q \mu$. Since

$$\mu(y * 0) + M(t_1, t_2) = \mu(y) + t_1 = t_y + t_1 < 1,$$

we have $(y * 0)_{M(t_1, t_2)} \notq \mu$, which is a contradiction. Next assume that $t_y > t_0$. Then $\mu(y) + (1 - t_0) = t_y + 1 - t_0 > 1$ and so $y_1 - t_0 q \mu$. Since

$$\mu(y * y) + (1 - t_0) = \mu(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

we get $(y * y)_{M(1 - t_0, 1 - t_0)} \notq \mu$. This is impossible. Therefore $\mu$ is constant on $X_0$.

**Theorem 3.12.** Let $\mu$ be a non-zero $(\alpha, \beta)$-fuzzy subalgebra of $X$, where $(\alpha, \beta)$ is one of the following:

- $(\in, q)$,
- $(q, \in)$,
- $(\in \lor q, q)$,
- $(\in \lor q, \in q)$,
- $(\in \lor q, \in)$.

Then $\mu = \chi_{X_0}$, the characteristic function of $X_0$. 

Assume that there exists \( x \in X_0 \) such that \( \mu(x) < 1 \). For \( \alpha = \varepsilon \), choose \( t \in (0, 1] \) such that \( t < M(1 - \mu(x), \mu(x)) \). Then \( \alpha \mu \) and \( 0_1 \alpha \mu \), but \( (x \ast 0)_{M(t, t)} = x_t \beta \mu \) where \( \beta = q \) or \( \beta = \varepsilon \land q \). This is a contradiction. Now let \( \alpha = q \). Then \( \alpha \mu \) and \( 0_1 \alpha \mu \), but \( (x \ast 0)_{M(t, 1)} = x_t \beta \mu \) for \( \beta = \varepsilon \) or \( \beta = \varepsilon \land q \), a contradiction. Finally let \( \alpha = \varepsilon \lor q \) and choose \( t \in (0, 1] \) such that \( x_t \in \mu \) but \( x_t \notin \mu \). Then \( \alpha \mu \) and \( 0_1 \alpha \mu \), but \( (x \ast 0)_{M(t, 1)} = x_t \beta \mu \) for \( \beta = q \) or \( \beta = \varepsilon \land q \). This is impossible. Note that \( \alpha \mu \) and \( 0_1 \alpha \mu \) but \( (x \ast 0)_{M(t, 1)} = x_t \in \mu \), a contradiction. Therefore \( \mu = \chi_{x_0} \).

**Theorem 3.13.** Let \( S \) be a subalgebra of \( X \) and let \( \mu \) be a fuzzy set in \( X \) such that

(i) \( \mu(x) = 0 \) for all \( x \in X \setminus S \),

(ii) \( \mu(x) \geq 0.5 \) for all \( x \in S \).

Then \( \mu \) is a \( (q, \in \lor q) \)-fuzzy subalgebra of \( X \).

**Proof.** Let \( x, y \in X \) and \( t_1, t_2 \in (0, 1] \) be such that \( x_{t_1} \in M \mu \) and \( y_{t_2} \in M \mu \), that is, \( \mu(x) + t_1 > 1 \) and \( \mu(y) + t_2 > 1 \). Then \( x \ast y \in S \) because if not then \( x \notin X \setminus S \) or \( y \notin X \setminus S \). Thus \( \mu(x) = 0 \) or \( \mu(y) = 0 \), and so \( t_1 > 1 \) or \( t_2 > 1 \). This is a contradiction. If \( M(t_1, t_2) > 0.5 \), then \( \mu(x \ast y) + M(t_1, t_2) > 1 \) and thus \( (x \ast y)_{M(t_1, t_2)} \in M \mu \). If \( M(t_1, t_2) \leq 0.5 \), then \( \mu(x \ast y) \geq 0.5 \geq M(t_1, t_2) \) and so \( (x \ast y)_{M(t_1, t_2)} \in M \mu \). Therefore \( (x \ast y)_{M(t_1, t_2)} \in \lor q \mu \). This completes the proof.

**Theorem 3.14.** Let \( \mu \) be a \( (q, \in \lor q) \)-fuzzy subalgebra of \( X \) such that \( \mu \) is not constant on \( X_0 \). Then there exists \( x \in X \) such that \( \mu(x) \geq 0.5 \). Moreover, \( \mu(x) \geq 0.5 \) for all \( x \in X \).

**Proof.** Assume that \( \mu(x) < 0.5 \) for all \( x \in X \). Since \( \mu \) is not constant on \( X_0 \), there exists \( x \in X_0 \) such that \( t_x = \mu(x) \neq \mu(0) = t_0 \). Then either \( t_0 < t_x \) or \( t_0 > t_x \). For the first case, choose \( \delta > 0.5 \) such that \( t_0 \lor \delta < t_x \lor \delta \). It follows that \( x_{t_0} \mu, \mu(x \ast x) = \mu(0) = y_0 < \delta = M(\delta, \delta) \), and \( \mu(x \ast x) + M(\delta, \delta) = \mu(0) + \delta = t_0 \lor \delta < 1 \) so that \( (x \ast x)_{M(\delta, \delta)} \notin \lor q \mu \). This is a contradiction. Now if \( t_0 > t_x \), we can choose \( \delta > 0.5 \) such that \( t_x \lor \delta < t_0 \lor \delta \). Then \( 0 \notin \mu \) and \( x_{t_0} \mu \), but \( (x \ast 0)_{M(t, \delta)} = x_t \notin \lor q \mu \) since \( \mu(x) < 0.5 \). This leads a contradiction. Therefore \( \mu(x) \geq 0.5 \) for some \( x \in X \). We now show that \( \mu(0) \geq 0.5 \). Assume that \( \mu(0) = t_0 < 0.5 \). Since there exists \( x \in X \) such that \( \mu(x) = t_x \geq 0.5 \), it follows that \( t_0 < t_x \). Choose \( t_1 > t_0 \) such that \( t_0 + t_1 < t_x + t_1 \). Then \( \mu(x) + t_1 = t_x + t_1 > 1 \), and so \( v_{t_1} \mu \). Now we get

\[
\mu(x \ast x) + M(t_1, t_1) = \mu(0) + t_1 = t_0 + t_1 < 1,
\]
For any subset $A$ fuzzy set Suppose that contradiction. Consequently, 

\[ \mu(x + x) = \mu(0) = t_0 < t_1 = M(t_1, t_1). \]

Hence $(x + x)_M(t_1, t_1) \in \vee q \mu$, a contradiction. Therefore $\mu(0) \geq 0.5$. Finally suppose that $t_x = \mu(x) < 0.5$ for some $x \in X_0$. Take $t > 0$ such that $t_x + t < 0.5$. Then $\mu(x) + 1 = t_x + 1 > 1$ and $\mu(0) + (0.5 + t) > 1$, which imply that $x_1 \in \mu$ and $0_{0.5+t} \in q \mu$. But $(x+0)_M(1,0.5+t) = x_{0.5+t} \in \vee q \mu$ since $\mu(x+0) = \mu(x) < 0.5 + t < M(1,0.5 + t)$ and

\[ \mu(x+0) + M(1,0.5 + t) = \mu(x) + 0.5 + t = t_x + 0.5 + t < 0.5 + 0.5 = 1. \]

This is a contradiction. Hence $\mu(x) \geq 0.5$ for all $x \in X_0$. This completes the proof.

**Theorem 3.15.** A fuzzy set $\mu$ in $X$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$ if and only if it satisfies:

\[ (\forall x, y \in X) \left( \mu(x \ast y) \geq M(\mu(x), \mu(y), 0.5) \right). \]

**Proof.** Suppose that $\mu$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$ and let $x, y \in X$. If $M(\mu(x), \mu(y)) < 0.5$, then $\mu(x \ast y) \geq M(\mu(x), \mu(y))$. For, assume that $\mu(x \ast y) < M(\mu(x), \mu(y))$ and choose $t$ such that $\mu(x \ast y) < t < M(\mu(x), \mu(y))$. Then $x_t \in \mu$ and $y_t \in \mu$ but $(x \ast y)_M(t, t) = (x \ast y)_t \in \vee q \mu$, a contradiction. Hence $\mu(x \ast y) \geq M(\mu(x), \mu(y))$ whenever $M(\mu(x), \mu(y)) < 0.5$. Now suppose that $M(\mu(x), \mu(y)) \geq 0.5$. Then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, which imply that

\[ (x \ast y)_{M(0.5,0.5)} = (x \ast y)_{0.5} \in \vee q \mu. \]

Thus $\mu(x \ast y) \geq 0.5$. Otherwise, $\mu(x \ast y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mu(x \ast y) \geq M(\mu(x), \mu(y), 0.5)$ for all $x, y \in X$. Conversely assume that (4) is valid. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. If $\mu(x \ast y) < M(t_1, t_2)$, then $M(\mu(x), \mu(y)) \geq 0.5$. Otherwise, we have

\[ \mu(x \ast y) \geq M(\mu(x), \mu(y), 0.5) \geq M(\mu(x), \mu(y)) \geq M(t_1, t_2), \]

a contradiction. It follows that

\[ \mu(x \ast y) + M(t_1, t_2) > 2\mu(x \ast y) \geq 2M(\mu(x), \mu(y), 0.5) = 1 \]

so that $(x \ast y)_{M(t_1, t_2)} q \mu$. Therefore $\mu$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$. \hfill \Box

**Theorem 3.16.** For any subset $S$ of $X$, the characteristic function $\chi_S$ of $S$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$ if and only if $S$ is a subalgebra of $X$. 

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Assume that $\chi_S$ is an $(\in, \vee, \lor)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. It follows that $(x * y)_1 = (x * y)_{M(1, 1)} \in \vee q \chi_S$ which yields $\chi_S(x * y) > 0$. Hence $xy \in S$, and thus $S$ is a subalgebra of $X$. Conversely if $S$ is a subalgebra of $X$, then $\chi_S$ is an $(\in, \vee, \lor)$-fuzzy subalgebra of $X$. It follows from Theorem 3.4 that $\chi_S$ is an $(\in, \vee, q)$-fuzzy subalgebra of $X$. 

**Theorem 3.17.** Let $\{\mu_i\} i \in \Lambda$ be a family of $(\in, \vee, q)$-fuzzy subalgebras of $X$. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \vee, q)$-fuzzy subalgebra of $X$.

**Proof.** Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_1 \in \mu$ and $y_2 \in \mu$. Assume that $(x * y)_{M(t_1, t_2)} \in \vee q \mu$. Then $\mu(x * y) < M(t_1, t_2)$ and $\mu(x * y) + M(t_1, t_2) \leq 1$, which imply that

\[ (5) \quad \mu(x * y) < 0.5. \]

Let $\Omega_1 := \{ i \in \Lambda | (x * y)_{M(t_1, t_2)} \in \mu_i \}$ and

\[ \Omega_2 := \{ i \in \Lambda | (x * y)_{M(t_1, t_2)} q \mu_i \} \cap \{ j \in \Lambda | (x * y)_{M(t_1, t_2)} \sqsupseteq \mu_j \}. \]

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then $(x * y)_{M(t_1, t_2)} \in \mu_i$ for all $i \in \Lambda$, that is, $\mu_i(x * y) \geq M(t_1, t_2)$ for all $i \in \Lambda$, which yields $\mu(x * y) \geq M(t_1, t_2)$. This is a contradiction. Hence $\Omega_2 \neq \emptyset$, and so for every $i \in \Omega_2$ we have $\mu_i(x * y) < M(t_1, t_2)$ and $\mu_i(x * y) + M(t_1, t_2) > 1$. It follows that $M(t_1, t_2) > 0.5$. Now $x_1 \in \mu$ implies $\mu(x) \geq t_1$ and thus $\mu_i(x) \geq \mu(x) \geq t_1 \geq M(t_1, t_2) > 0.5$ for all $i \in \Lambda$. Similarly we get $\mu_i(y) > 0.5$ for all $i \in \Lambda$. Next suppose that $t := \mu_i(x * y) < 0.5$. Taking $t < r < 0.5$, we get $x_r \in \mu_i$ and $y_r \in \mu_i$, but $(x * y)_{M(r, r)} = (x * y)_r \in \vee q \mu_i$. This contradicts that $\mu_i$ is an $(\in, \vee, q)$-fuzzy subalgebra of $X$. Hence $\mu_i(x * y) \geq 0.5$ for all $i \in \Lambda$, and so $\mu(x * y) \geq 0.5$ which contradicts (5). Therefore $(x * y)_{M(t_1, t_2)} \in \vee q \mu$ and consequently $\mu$ is an $(\in, \vee, q)$-fuzzy subalgebra of $X$. 

References

On $(\alpha, \beta)$-fuzzy subalgebras of $BCK/BCI$-algebras

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