GENERALIZED FRÉCHET-URYSOHN SPACES

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Abstract. In this paper, we introduce some new properties of a topological space which are respectively generalizations of Fréchet-Urysohn property. We show that countably AP property is a sufficient condition for a space being countable tightness, sequential, weakly first countable and symmetrizable to be ACP, Fréchet-Urysohn, first countable and semi-metrizable, respectively. We also prove that countable compactness is a sufficient condition for a countably AP space to be countably Fréchet-Urysohn. We then show that a countably compact space satisfying one of the properties mentioned here is sequentially compact. And we show that a countably compact and countably AP space is maximal countably compact if and only if it is Fréchet-Urysohn. We finally obtain a sufficient condition for the ACP closure operator \([\cdot]_{ACP}\) to be a Kuratowski topological closure operator and related results.

1. Introduction and preliminaries

All spaces are assumed to be Hausdorff. Our terminology is standard and follows [2] and [5]. Let \(X\) be a topological space and let \(c\) denote the closure operator on the space \(X\). Let \(\mathbb{N}\) denote the set of all natural numbers and \((x_n|n \in \mathbb{N})(\text{briefly } (x_n))\) a sequence of points of a set. The following functions \([\cdot]_{seq}\), \([\cdot]_{AP}\), and \([\cdot]_{ACP}\) of the power set \(\mathcal{P}(X)\) of \(X\) to \(\mathcal{P}(X)\) itself defined by for each subset \(A\) of \(X\),

\[
[A]_{seq} = \{x \in X : (x_n)\text{ converges to }x\text{ in }X\text{ for some sequence } (x_n)\text{ of points of }A\},
\]

\[
[A]_{AP} = A \cup \{x \in c(A) - A : c(F) = F' \cup \{x\}\text{ for some subset }F\text{ of }A\},
\]

\[
[A]_{ACP} = A \cup \{x \in c(A) - A : c(F) = F' \cup \{x\}\text{ for some countable subset }F\text{ of }A\}
\]

are called the sequential closure operator on \(X\) [2], the AP closure operator on \(X\) [15], and the ACP closure operator on \(X\), respectively. It is well known that the sequential closure operator \([\cdot]_{seq}\) satisfies the Kuratowski topological closure axioms except for idempotency in general (see [2]). We see easily that
for each subset \( A \) of \( X \),
\[
A \subset [A]_{\text{seq}} \subset [A]_{\text{ACP}} \subset [A]_{\text{AP}} \subset c(A),
\]
for each countable subset \( A \) of \( X \), \([A]_{\text{AP}} = [A]_{\text{ACP}}\), and \([A]_{\text{AP}}\) and \([A]_{\text{ACP}}\) do not satisfy the Kuratowski topological closure axioms in general.

Let us recall some properties and introduce new three properties of a topological space \( X \).

1. **Fréchet-Urysohn** [2] (also called Fréchet [6]) : for each subset \( A \) of \( X \), \([A]_{\text{seq}} = c(A)\).
2. **sequential** [6] : for each subset \( A \) of \( X \) which is not closed in \( X \), \([A]_{\text{seq}} - A \neq \emptyset\).
3. **countable tightness** [1] (also called determined by countable subsets [10], [12]) : for each subset \( A \subset X \) and each \( x \in c(A) \), there exists a countable subset \( B \) of \( A \) such that \( x \in c(B) \).
4. **countably Fréchet-Urysohn** [9] : for each countable subset \( A \) of \( X \), \([A]_{\text{seq}} = c(A)\).
5. **AP** (standing for Approximation by Points) [15] (also called Whyburn [11]) : for each subset \( A \) of \( X \), \([A]_{\text{AP}} = c(A)\).
6. **WAP** (standing for Weak Approximation by Points) [15] (also called weakly Whyburn [11]) : for each subset \( A \) of \( X \) which is not closed in \( X \), \([A]_{\text{AP}} - A \neq \emptyset\).
7. **countably AP** : for each countable subset \( A \) of \( X \), \([A]_{\text{AP}} = c(A)\).
8. **ACP** (standing for Approximation by Countable Points) : for each subset \( A \) of \( X \), \([A]_{\text{ACP}} = c(A)\).
9. **WACP** (standing for Weak Approximation by Countable Points) : for each subset \( A \) of \( X \) which is not closed in \( X \), \([A]_{\text{ACP}} - A \neq \emptyset\).

From definitions and Hausdorffness of \( X \), one easily know that the following diagram except for \(*\) exhibits the general relationships among the properties mentioned above. No arrows may be reversed as shown by Example below (see [1, 2, 3, 4, 6, 8, 9, 10, 11, 15]).

![Diagram of relationships among topological properties](image-url)

We begin by showing some examples related to the new three properties.

**Example 1.1.** (1) Let \( X = \{(0, 0)\} \cup \mathbb{N} \times \mathbb{N} \). We define a topology \( \tau \) on \( X \) by for each \((m, n) \in X - \{(0, 0)\} \), \(\{(m, n)\} \in \tau \) and \((0, 0) \in U \in \tau \) if and only...
if for all but a finite number of integers \( m \), the sets \( \{ n \in \mathbb{N} : (m, n) \notin U \} \) are each finite. Thus each point \((m, n) \in X - \{(0, 0)\}\) is isolated and each open neighborhood of \((0, 0)\) contains all but a finite number of points in each of all but a finite number of columns (see Arens-Fort space in [14]). Then it is clear that the space \( X \) is Hausdorff and there is a unique non-isolated point \((0, 0)\) in \( X \). In fact, \( X \) is normal. Note that any space with a unique non-isolated point is AP ([15, Proposition 2.1(10)]). It follows that since \( X \) is countable, \( X \) is ACP and hence countable tightness, WACP, and countably AP. Clearly, \( (0, 0) \) is AP and hence WAP and countably AP. But, it is neither countable Fréchet-Urysohn, nor Fréchet-Urysohn.

(2) Let \( X = \{ z \} \cup \mathbb{R} \), where \( \mathbb{R} \) is the set of all real numbers. We define a topology \( \tau \) on \( X \) by for each \( x \in \mathbb{R} \), \( \{ x \} \in \tau \) and \( z \in U \in \tau \) if and only if \( \mathbb{R} - U \) is countable. Clearly, \( X \) is Hausdorff and \( z \) is a unique non-isolated point in \( X \). Thus \( X \) is AP and hence WAP and countably AP. But, it is neither countable tightness, sequential, WACP, nor ACP. For, \( z \in c(\mathbb{R}) \), but there does not exist a countable subset \( C \) of \( \mathbb{R} \) such that \( z \in c(C) \) since every countable subset of \( \mathbb{R} \) is closed in \( X \).

(3) Let \( \mathbb{R} \) be the set of real numbers, \( \tau_1 \) the usual topology on \( \mathbb{R} \) and \( \tau_2 \) the topology of countable complements on \( \mathbb{R} \). We define \( \tau \) to be the smallest topology on \( \mathbb{R} \) generated by \( \tau_1 \cup \tau_2 \). Then a set \( U \) is open in the space \( (\mathbb{R}, \tau) \) if and only if \( U = O - K \) where \( O \in \tau_1 \) and \( K \) is a countable subset of \( \mathbb{R} \) (see Countable Complement Extension Topology in [14]). Clearly, the space \( (\mathbb{R}, \tau) \) is Hausdorff and every countable subset of \( \mathbb{R} \) is closed in \( (\mathbb{R}, \tau) \). It is easy to check that if every countable subset of a topological space \( X \) is closed, then \( X \) is countably AP. Thus \( (\mathbb{R}, \tau) \) is a countably AP space.

We now show that the space \( \mathbb{R} \) is not AP. Suppose \( \mathbb{R} \) is AP and let \( A = [0, 1] - Q \), where \([0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \) and \( Q \) is the set of all rational numbers. Then since \( \mathbb{R} \) is AP and \( 0 \in c(A) - A \), there exists a subset \( B \) of \( A \) such that \( c(B) = B \cup \{0\} \). Since every countable subset of \( \mathbb{R} \) is closed, clearly the set \( B \) must be uncountable. It follows that for each \( x \in (0, 1] \cap Q \), \( x \notin c(B) \) and hence there are \( \epsilon_x > 0 \) and a countable subset \( K_x \) of \( \mathbb{R} \) such that \( ((x - \epsilon_x, x + \epsilon_x) - K_x) \cap B = \emptyset \). We then have that

\[
\bigcup \{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in (0, 1] \cap Q \} \cap B = \emptyset.
\]

Clearly,

\[
\bigcup \{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in (0, 1] \cap Q \} \supset A - \bigcup \{K_x : x \in (0, 1] \cap Q \}.
\]

Thus,

\[
(A - \bigcup \{K_x : x \in (0, 1] \cap Q \}) \cap B = \emptyset,
\]

and so \( B \subset \bigcup \{K_x : x \in (0, 1] \cap Q \} \); that is, an uncountable set \( B \) is a subset of a countable set \( \bigcup \{K_x : x \in (0, 1] \cap Q \} \), which is a contradiction.

(4) Let \( X \) be the set containing of pairwise distinct objects of the following three types : points \( x_{mn} \) where \( m, n \in \mathbb{N} \), points \( y_n \) where \( n \in \mathbb{N} \) and a point
We set $V_k(y_n) = \{y_n\} \cup \{x_{mn} : m \geq k\}$ for each $k \in \mathbb{N}$ and let $\gamma$ denote the set of subsets $W$ of $X$ such that $z \in W$ and there exists a positive integer $p$ such that $V_1(y_n) - W$ is finite and $y_n \in W$ for all $n \geq p$. The collection 
\[
\{\{x_{mn} : m, n \in \mathbb{N}\} \cup \gamma \cup \{V_k(y_n) : k, n \in \mathbb{N}\}\}
\] is a base for a topology on $X$. It is clear that the space $X$ with the topology generated by the base is Hausdorff sequential [2, p.13, Example 13] and hence WACP and WAP. Let $Y = \{x_{mn} : m, n \in \mathbb{N}\}$. Then, $z \in c(Y)$. We know that for each subset $F$ of $Y$ with $z \in c(F)$, $\{y_n : n \in \mathbb{N}\} \cap F$ is infinite. Hence, there does not exist a subset $F$ of $Y$ such that $c(F) = F \cup \{z\}$, and thus $X$ is neither AP nor ACP.

(5) The space of ordinals $X = [0, \omega_1]$, where $\omega_1$ is the first uncountable ordinal, is compact Hausdorff, WAP (see [14, p.70, 43(14)] and [15, Theorem 2.7]), and countably Fréchet-Urysohn ([9, Example (2)]) and hence countably AP. But, it is neither sequential ([6]) nor AP ([15, Corollary 2.10]).

Here we observe the implication * in the diagram above.

**Theorem 1.2.** Every WACP space is countable tightness.

**Proof.** It is well-known that a topological space $X$ is countable tightness if and only if for each non-closed subset $A$ of $X$, there are $x \in c(A) - A$ and a sequence $(x_n)$ of points of $A$ such that $(x_n)$ accumulates at $x$ (see [12, Proposition 2.2]). Let $X$ be a WACP space and $A$ a non-closed subset of $X$. Then since $X$ is WACP, $[A]_{ACP} - A \neq \emptyset$, and hence there are $x \in [A]_{ACP} - A$ and a countable subset $C$ of $A$ such that $c(C) = C \cup \{x\}$. By Hausdorffness of $X$, we see that $C$ is countably infinite. Let $C = \{x_n : n \in \mathbb{N}\}$. Then since $x \in c(C) - C$, for each open set $U_x$ in $X$ containing $x$, $C \cap U_x$ is infinite. It follows that the sequence $(x_n)$ accumulates at $x$. Thus we have that there are $x \in [A]_{ACP} - A \subset c(A) - A$ and a sequence $(x_n)$ of points of $A$ such that $(x_n)$ accumulates at $x$. Therefore $X$ is countable tightness. \qed

In [4, Theorem 2.1], A Bella and I. V. Yaschenko showed that there is a countable non WAP space. Since every countable space is countable tightness and every WACP space is WAP, we have that there is a countable tightness non WACP space and hence the reverse of * is not true in general.

In [3, Proposition 3], A. Bella showed that a countably compact and WAP space is sequentially compact and in [15, Proposition 2.1(12) and Theorem 2.2], V. V. Tkachuk and I. V. Yaschenko showed that the space $\beta N - N$ is not WAP (and hence $\beta N$ is neither WAP) and a countably compact and AP space is Fréchet-Urysohn.

In particular, in [7, Theorem 9.6](also in [13, Theorem 1.10] and [8, Theorem 2.2]), the authors showed a well-known and useful theorem that the following statements are equivalent:

(1) $X$ is semi-metrizable.
(2) $X$ is symmetrizable and first countable.
(3) $X$ is symmetrizable and Fréchet-Urysohn.

Also, J. E. Vaughan in [16, p.590, 5.3] and S. P. Franklin in [6, Proposition 1.10] proved that a countably compact and sequential space is sequentially compact. In [2, p.58, Proposition 3], A. V. Arhangel’skii and L. S. Pontryagin showed that a compact and Fréchet-Urysohn space is sequentially compact.

Recently, in [9, Lemma 2.1], the author showed that in a countably Fréchet-Urysohn space $X$, the sequential closure operator $[\cdot]_{\text{seq}}$ on $X$ satisfies the Kuratowski topological closure axioms and the space $X$ endowed with the topology induced by $[\cdot]_{\text{seq}}$ is Fréchet-Urysohn. And in [9, Theorem 2.4(1)], he also showed that a sequentially compact and countably Fréchet-Urysohn space is Fréchet-Urysohn if and only if it is maximal sequentially compact.

In this paper, we introduce some new properties of a topological space which are respectively generalizations of Fréchet-Urysohn property. We then give some examples and investigate the relationships among the properties. We prove that countably AP property is a sufficient condition for a space being countable tightness, sequential, weakly first countable and symmetrizable to be ACP, Fréchet-Urysohn, first countable and semi-metrizable, respectively. We also prove that countable compactness is a sufficient condition for a countably AP space to be countably Fréchet-Urysohn. We then show that a countably compact space satisfying one of the properties mentioned above except for countable tightness is sequentially compact. And we show that a countably compact and countably AP space is maximal countably compact if and only if it is Fréchet-Urysohn. Finally, we show that if a topological space $X$ is countably AP, then the ACP closure operator $[\cdot]_{\text{ACP}}$ on $X$ satisfies the Kuratowski topological closure axioms and the space $X$ endowed with the topology induced by $[\cdot]_{\text{ACP}}$ is ACP. Moreover, if $X$ is a countably compact and countably AP space, then the ACP expansion of $X$ obtained above is Fréchet-Urysohn.

2. Results

We now show the relationships among the properties.

**Theorem 2.1.**
(1) Every countably Fréchet-Urysohn and countable tightness space is Fréchet-Urysohn.

(2) Every countably AP and sequential space is Fréchet-Urysohn.

(3) Every countably AP and countable tightness space is ACP.

(4) Every WAP and countable tightness space is WACP.

**Proof.** (1) See [10, Proposition 8.7].

(2) Let $X$ be a countably AP and sequential space, $A$ a subset of $X$ and $x \in c(A)$. Then since $X$ is sequential and hence countable tightness, there is a countable subset $C$ of $A$ such that $x \in c(C)$. Since $X$ is countably AP and $C$ is countable, there is a subset $F$ of $C$ such that $c(F) = F \cup \{x\}$; i.e., $\{x\} = c(F) - F$. Since $X$ is sequential, there exists a sequence $(x_n)$ of points...
of $F$ such that $(x_n)$ converges to $x$. Thus $x \in [F]_{\text{seq}} \subset [C]_{\text{seq}} \subset [A]_{\text{seq}}$, and so $X$ is Fréchet-Urysohn.

(3) Let $X$ be a countably AP and countable tightness space, $A$ a subset of $X$ and $x \in c(A)$. Then since $X$ is countable tightness, there exists a countable subset $C$ of $A$ such that $x \in c(C)$. Since $X$ is countably AP and $C$ is countable, there exists a subset $F$ of $C$ such that $c(F) = F \cup \{x\}$, and hence $x \in [A]_{ACP}$. Thus $c(A) = [A]_{ACP}$, and so $X$ is ACP.

(4) Let $X$ be a WAP and countable tightness space and $A$ a non-closed subset of $X$. Then since $X$ is countable tightness, there exist $x \in c(A) - A$ and a sequence $(x_n)$ of points of $A$ such that $(x_n)$ accumulates at $x$. Clearly, $x \in c(\{x_n : n \in \mathbb{N}\})$. Since $X$ is WAP, there exists a subset $F$ of $\{x_n : n \in \mathbb{N}\}$ such that $c(F) = F \cup \{x\}$. Hence we have that there exist $x \in c(A) - A$ and a countable subset $F$ of $A$ such that $c(F) = F \cup \{x\}$, and thus $X$ is WACP. □

By Theorems 1.2 and 2.1 above, we have immediately the following corollary.

**Corollary 2.2.**

(1) Every WACP and countably AP space is ACP.

(2) Every AP and countable tightness space is ACP.

(3) Every countably AP and countable tightness space is AP.

(4) Every countable tightness and countably AP space is WACP.

(5) Every countably Fréchet-Urysohn and WACP space is Fréchet-Urysohn.

(6) Every sequential and AP space is Fréchet-Urysohn (see [15, Proposition 2.1(6)].

Note that by Example 1.1(1), we see that every countably AP and countable tightness space need not be countably Fréchet-Urysohn in general.

Recall that a topological space $X$ is called weakly first countable [7] (also called $g$-first countable [13]) if for each $x \in X$, there exists a family $\{B(x, n) : n \in \mathbb{N}\}$ of subsets of $X$ such that the following conditions are satisfied:

(i) $x \in B(x, n + 1) \subset B(x, n)$ for all $n \in \mathbb{N}$,

(ii) a subset $U$ of $X$ is open if and only if for every $x \in U$ there exists an $n \in \mathbb{N}$ such that $B(x, n) \subset U$.

Such a family $\{B(x, n) : n \in \mathbb{N}\}$ is called a weak base at $x$.

A topological space $X$ is called symmetrizable [7] if there exists a symmetric ($=$ a metric except for the triangle inequality) $d$ on $X$ satisfying the following condition: a subset $U$ of $X$ is open if and only if for every $x \in U$ there is a positive real number $r$ such that $B(x, r) \subset U$, where $B(x, r)$ denotes the set $\{y \in X : d(x, y) < r\}$. A space $X$ is semi-metrizable [7] if and only if there exists a symmetric $d$ on $X$ such that for each $x \in X$, the family $\{B(x, r) : r > 0\}$ forms a (not necessarily open) neighborhood base at $x$.

By definitions and the first diagram above, we have that the following second diagram indicates the general nature of these properties above (see [7, 8, 13]).
Theorem 2.3. If a space $X$ satisfies one of the properties in the second column of the second diagram above and is also countably AP, then it satisfies the corresponding property in the first column.

Proof. In Theorem 2.1(2) and (3), we have shown that every countable tightness and countably AP space is ACP and every sequential and countably AP space is Fréchet-Urysohn.

Let $X$ be a weakly first countable and countably AP space. Then since $X$ is weakly first countable, for each $x \in X$, there is a weak base $\{B(x, n) : n \in \mathbb{N}\}$ at $x$. To prove this, it is sufficient to show that for each $n \in \mathbb{N}$, $B(x, n)$ is a neighborhood of $x$ in the space $X$. Suppose on the contrary that there are $x \in X$ and $n \in \mathbb{N}$ such that $x \notin \text{int}(B(x, n))$, where $\text{int}(B(x, n))$ is the interior of $B(x, n)$ in $X$. Then, clearly, $x \in c(X - B(x, n))$. Since $X$ is weakly first countable and countably AP, it is sequential and countably AP. By Theorem 2.1(2), $X$ is Fréchet-Urysohn and hence AP. It follows that there exists a subset $Y$ of $X - B(x, n)$ such that $c(Y) = Y \cup \{x\}$. Since $X - c(Y)$ is open and $B(x, n) \subset ((X - c(Y)) \cup \{x\})$, by (ii) of the definition of weak first countability, we have that the set $(X - c(Y)) \cup \{x\}$ is open in $X$ containing $x$ and $Y \cap ((X - c(Y)) \cup \{x\}) = \emptyset$, which is a contradiction. Thus we have that a weakly first countable and countably AP space is first countable.

Finally, by the above result, it follows that every symmetrizable and countably AP space is first countable and hence semi-metrizable. \qed

From Theorem 2.3, we have immediately the following corollaries and hence we omit the proofs.

Corollary 2.4. Let $X$ be a sequential space. Then the following statements are equivalent:

1. $X$ is Fréchet-Urysohn.
2. $X$ is ACP.
3. $X$ is countably AP.

Corollary 2.5. Let $X$ be a weakly first countable space. Then the following statements are equivalent:
Corollary 2.6. Let \( X \) be a symmetrizable space. Then the following statements are equivalent:

1. \( X \) is semi-metrizable.
2. \( X \) is first countable.
3. \( X \) is Fréchet-Urysohn.
4. \( X \) is ACP.
5. \( X \) is countably AP.

We also obtain the result of [7, Theorem 9.6] (also [13, Theorem 1.10] and [8, Theorem 2.2]) as a corollary.

By Example 1.1(5), we know that a WAP and countably AP space need not be an AP space. Hence, in Theorem 2.3 above, we cannot replace “ACP \( \rightarrow \) countable tightness” by “AP \( \rightarrow \) WAP”.

We recall that a topological space \( X \) is countably compact if and only if every countable open cover of \( X \) has a finite subcover; equivalently, every sequence of points of \( X \) has an accumulation point.

We are going to prove that countably compactness is a sufficient condition for a countably AP to be countably Fréchet-Urysohn.

Theorem 2.7. Every countably compact and countably AP space is countably Fréchet-Urysohn.

Proof. In [15, Theorem 2.2], V. V. Tkachuk and I. V. Yaschenko showed that a countably compact and AP space is Fréchet-Urysohn. It can be proved using the very similar arguments of the proof of [15, Theorem 2.2]. Hence we omit the proof. \( \square \)

Remarks 2.8. (1) From Example 1.1(5), we know that a compact and WAP space need not be sequential in general.

(2) Still there is a very natural question left open: Is every countably compact (or compact) and WACP space sequential?

Recall that a topological space \( X \) is sequentially compact if and only if every sequence of points of \( X \) has a convergent subsequence. It is obvious that every sequentially compact space is countably compact, but the reverse is not true in general.

Next, we show that a countably compact space satisfying one of the properties mentioned above except for countable tightness is sequentially compact.

Theorem 2.9. Every countably compact and countably AP space is sequentially compact.
Proof. Let $X$ be a countably compact and countably AP space and let $(x_n)$ be a sequence of points of $X$. Then since $X$ is countably compact, $(x_n)$ has an accumulation point. Let $x$ be an accumulation point of $(x_n)$ in $X$. Clearly, $x \in c\{x_n : n \in \mathbb{N}\}$, where $\{x_n : n \in \mathbb{N}\}$ is the range of $(x_n)$. By Theorem 2.7, $X$ is countably Fréchet-Urysohn and hence $c\{x_n : n \in \mathbb{N}\} = [\{x_n : n \in \mathbb{N}\}]_{seq}$. It follows that there exists a sequence $(y_n)$ of points of $\{x_n : n \in \mathbb{N}\}$ such that $(y_n)$ converges to $x$. Set $y_n = x_{\mu(n)}$ for each $n \in \mathbb{N}$. Note that $(y_n)$ need not be a subsequence of $(x_n)$ in general. We now construct a sequence $(x_{\phi(n)})$ as follows: Let $\phi(1) = \mu(1)$, $\phi(2) = \text{the first}(\text{least})$ element of $\{\mu(k) | \phi(1) < \mu(k), k \in \mathbb{N}\}$ and $\phi(3) = \text{the first element of } \{\mu(k) | \phi(2) < \mu(k) \text{ and } p < k\}$, where $p$ is the number satisfying $x_{\phi(2)} = x_{\mu(p)} = y_p$. Assume that for $k \in \mathbb{N}$, $\phi(1) < \phi(2) < \phi(3) < \cdots < \phi(k)$ have been defined, let $\phi(k + 1) = \text{the first element of } \{\mu(k) | \phi(k) < \mu(k) \text{ and } p < k\}$, where $p$ is the number satisfying $x_{\phi(k)} = x_{\mu(p)} = y_p$. Then we obtain, by Induction, a sequence $(x_{\phi(n)})$. It is obvious that the sequence $(x_{\phi(n)})$ is a subsequence of $(x_n)$ and also a subsequence of $(y_n)$. Since $(y_n)$ converges to $x$, $(x_{\phi(n)})$ converges to $x$. Hence we have shown that there exists a subsequence $(x_{\phi(n)})$ of $(x_n)$ which converges to $x$. Thus $X$ is sequentially compact. $\Box$

**Corollary 2.10.** A countably compact space which satisfies one of the properties mentioned above except for countable tightness is sequentially compact.

**Proof.** This follows directly from [3, Proposition 3] and Theorem 2.9. $\Box$

We obtain immediately the results of [16, p.590, 5.3], [6, Proposition 1.10], and [2, p.53, Proposition 3] from Corollary 2.10.

**Lemma 2.11.** ([9, Lemma 2.1]) If $(X, T)$ is a countably Fréchet-Urysohn space, then the sequential closure operator $[\cdot]_{seq}$ on $X$ satisfies the Kuratowski topological closure operator axioms and $(X, [\cdot]_{seq})$ is a Fréchet-Urysohn space, where $[\cdot]_{seq}$ is the topology for $X$ induced by the closure operator $[\cdot]_{seq}$ on $X$.

Note that $(X, [\cdot]_{seq})$ is a Fréchet-Urysohn expansion of a countably Fréchet-Urysohn space $(X, T)$.

**Lemma 2.12.** (1) If $(X, T)$ and $(X, T^*)$ are sequentially compact spaces with $T \subset T^*$, then for each sequence $(x_n)$ of points of $X$, $(x_n)$ converges to $x$ in $(X, T)$ if and only if $(x_n)$ converges to $x$ in $(X, T^*)$.

(2) Let $(X, T)$ be a countably Fréchet-Urysohn space. Then, $(X, T)$ is countably compact (sequentially compact) if and only if the Fréchet-Urysohn space $(X, [\cdot]_{seq})$ obtained in Lemma 2.11 is countably compact (resp. sequentially compact).

**Proof.** (1) See [9, Lemma 2.3].

(2) Note that a topological space $X$ is countably compact if and only if every countably infinite subset of $X$ has at least one cluster point. Let $F$ be a countably infinite subset of $X$. Since $(X, T)$ is countably Fréchet-Urysohn and
$F$ is countable, we have $c(F) = [F]_{\text{seq}} = c_{T_{\text{seq}}} (F)$, where $c_{T_{\text{seq}}} (F)$ denotes the closure of $F$ in $(X, T_{\text{seq}})$. By countable compactness of $X$, $F \subseteq c(F)$, and hence $F \subseteq c_{T_{\text{seq}}} (F)$. Thus $(X, T_{\text{seq}})$ is countably compact.

Conversely, let $F$ be a countably infinite subset of $X$. Since $(X, T_{\text{seq}})$ is countably compact and Fréchet-Urysohn, we have that $F \not\subseteq c_{T_{\text{seq}}} (F)$ and $c(F) = [F]_{\text{seq}} = c_{T_{\text{seq}}} (F)$, and hence $F \not\subseteq c(F)$. Thus $(X, T)$ is countably compact.

By Theorem 2.9, for sequential compactness, it is trivial. □

**Theorem 2.13.** Let $(X, T)$ be a countably compact and countably AP space. Then, $X$ is maximal countably compact if and only if $X$ is Fréchet-Urysohn.

**Proof.** Suppose $(X, T)$ is not Fréchet-Urysohn. Since $(X, T)$ is countably compact and countably AP, $(X, T)$ is countably Fréchet-Urysohn by Theorem 2.7. Hence, by Lemma 2.11, there is the Fréchet-Urysohn expansion $(X, T_{\text{seq}})$ of $(X, T)$. Since $(X, T)$ is not Fréchet-Urysohn, it follows clearly $T \not\subseteq T_{\text{seq}}$. But, by maximal countable compactness and Lemma 2.12(2), $T = T_{\text{seq}}$. This is a contradiction.

Conversely, suppose $(X, T)$ is not maximal countably compact. Then since $(X, T)$ is Fréchet-Urysohn, by Corollary 2.10, $(X, T)$ is not maximal sequentially compact and hence there exists a sequentially compact space $(X, T^*)$ such that $T \not\subseteq T^*$. Let $U \in T^* - T$. Clearly, $X - U$ is not closed in $(X, T)$, and so $c(X - U) - (X - U) \neq \emptyset$, where $c$ is the closure operator on $(X, T)$. Let $p \in c(X - U) - (X - U)$. Then since $X$ is Fréchet-Urysohn, $p \in [X - U]_{\text{seq}} = c(X - U)$ and hence there exists a sequence $(x_n)$ of points of $X - U$ such that $(x_n)$ converges to $p$ in $(X, T)$. By Lemma 2.12(1), $(x_n)$ converges to $p$ in $(X, T^*)$. Thus,

$$p \in c_{T^*} \{x_n : n \in \mathbb{N}\} \subseteq c_{T^*} (X - U) = X - U,$$

which is a contradiction. □

**Corollary 2.14.** ([9, Theorem 2.4(1)]) Let $X$ be a sequentially compact and countably Fréchet-Urysohn space. Then, $X$ is maximal sequentially compact if and only if $X$ is Fréchet-Urysohn.

**Proof.** This follows immediately from Theorems 2.9 and 2.13. □

**Corollary 2.15.** Let $X$ be a countably compact and countably AP space. Then, $X$ is maximal countably compact if and only if $X$ is countable tightness.

**Proof.** This follows from Theorems 2.1(1), 2.7, and 2.13. □

It is easy to check that the ACP closure operator $[\cdot]_{\text{ACP}}$ on a topological space $X$ is not a Kuratowski topological closure operator on $X$ in general. We finally obtain a sufficient condition for $[\cdot]_{\text{ACP}}$ to be a Kuratowski topological closure operator and related results.
Theorem 2.16. If \((X, T)\) is a countably AP space, then the ACP closure operator \([\cdot]_{ACP}\) on \(X\) satisfies the Kuratowski topological closure axioms and \((X, T_{\lceil}^{\cdot}_{ACP})\) is an ACP space, where \(T_{\lceil}^{\cdot}_{ACP}\) is the topology for \(X\) induced by \([\cdot]_{ACP}\). Moreover, if \((X, T)\) is a countably compact and countably AP space, then \((X, T_{\lceil}^{\cdot}_{ACP})\) is a countably compact and ACP space and hence Fréchet-Urysohn.

Proof. First, we show that the ACP closure operator \([\cdot]_{ACP}\) on \(X\) satisfies the Kuratowski topological closure axioms. By the definition of \([\cdot]_{ACP}\), it is obvious that \([X]_{ACP} = X\), \([\emptyset]_{ACP} = \emptyset\) and for each subsets \(A\) and \(B\) of \(X\), \([A]_{ACP} \cup [B]_{ACP} \subseteq [A \cup B]_{ACP}\). Let \(x \in [A \cup B]_{ACP}\). Then there exists a countable subset \(F\) of \(A \cup B\) such that \(c(F) = F \cup \{x\}\). Put \(F_A = A \cap F\) and \(F_B = B \cap F\). Then since

\[
x \in c(F) = c(F_A \cup F_B) = c(F_A) \cup c(F_B),
\]

\(x \in c(F_A)\) or \(x \in c(F_B)\). Without loss of generality, assume \(x \in c(F_A)\). Since \(X\) is countably AP and \(F_A\) is countable, there exists a subset \(G\) of \(F_A\) such that \(c(G) = G \cup \{x\}\) and hence \(x \in [A]_{ACP}\). Thus, for each subsets \(A\) and \(B\) of \(X\), \([A]_{ACP} \cup [B]_{ACP} = [A \cup B]_{ACP}\). Hence, it remains to prove that for each subset \(A\) of \(X\), \([A]_{ACP} = [[A]_{ACP}]_{ACP}\). Clearly, \([A]_{ACP} \subseteq [[A]_{ACP}]_{ACP}\). Conversely, let \(x \in [[A]_{ACP}]_{ACP}\). Then there exists a countable subset \(F\) of \([A]_{ACP}\) such that \(c(F) = F \cup \{x\}\). Since \(F\) is countable, \(F - A\) is at most countable. Let \(F - A = \{y_n : n \in \mathbb{N}\}\). Then for each \(n \in \mathbb{N}\), since \(y_n \in [A]_{ACP} - A\), there exists a countable subset \(A_{y_n}\) of \(A\) such that \(c(A_{y_n}) = A_{y_n} \cup \{y_n\}\). Clearly, \(A \cap F = F - \{y_n : n \in \mathbb{N}\}\) and \(y_n \in c(A_{y_n}) \subseteq c(\bigcup_{n \in \mathbb{N}} A_{y_n})\). We have that

\[
F = (A \cap F) \cup (F - A) = (A \cap F) \cup \{y_n : n \in \mathbb{N}\}
\]

\[
\subseteq (A \cap F) \cup c(\bigcup_{n \in \mathbb{N}} A_{y_n}) \subseteq c((A \cap F) \cup \bigcup_{n \in \mathbb{N}} A_{y_n}), \]

and hence \(x \in c(F) \subseteq c((A \cap F) \cup \bigcup_{n \in \mathbb{N}} A_{y_n})\). Since \(X\) is countably AP and \((A \cap F) \cup \bigcup_{n \in \mathbb{N}} A_{y_n}\) is countable, there exists a subset \(G\) of \((A \cap F) \cup \bigcup_{n \in \mathbb{N}} A_{y_n}\) such that \(c(G) = G \cup \{x\}\), and thus \(x \in [A]_{ACP}\). Therefore, \([\cdot]_{ACP}\) satisfies the Kuratowski topological closure axioms.

Second, we show that \((X, T_{\lceil}^{\cdot}_{ACP})\) is an ACP space. Let \(A\) be a subset of \(X\). Then by the definitions, it is obvious \([A]_{ACP} - T_{\lceil}^{\cdot}_{ACP} \subseteq cT_{\lceil}^{\cdot}_{ACP} (A)\), where \([A]_{ACP} - T_{\lceil}^{\cdot}_{ACP} \) and \(cT_{\lceil}^{\cdot}_{ACP} (A)\) are the ACP closure of \(A\) and the closure of \(A\) in \((X, T_{\lceil}^{\cdot}_{ACP})\), respectively. Hence it is sufficient to show that \(cT_{\lceil}^{\cdot}_{ACP} (A) \subseteq [A]_{ACP} - T_{\lceil}^{\cdot}_{ACP}\). Let \(x \in cT_{\lceil}^{\cdot}_{ACP} (A)\). Since \(cT_{\lceil}^{\cdot}_{ACP} (A) = [A]_{ACP}, x \in [A]_{ACP}\) and hence there exists a countable subset \(F\) of \(A\) such that \(c(F) = F \cup \{x\}\). Note that in a countably AP space \(X\), for each countable subset \(F\) of \(X\), \(c(F) = [F]_{ACP} = [F]_{AP}\). Thus \([F]_{ACP} = F \cup \{x\}\), and so \(x \in [A]_{ACP} - T_{\lceil}^{\cdot}_{ACP}\). Therefore, \((X, T_{\lceil}^{\cdot}_{ACP})\) is an ACP space.
Finally we show that \((X, T_{\text{ACP}})\) is a Fréchet-Urysohn space. It is trivial that \((X, T_{\text{ACP}})\) is Hausdorff since \((X, T)\) is Hausdorff and \(T \subset T_{\text{ACP}}\). By [15, Theorem 2.2], we have known that every countably compact and AP (ACP) space is Fréchet-Urysohn. Hence, it is sufficient to prove that \((X, T_{\text{ACP}})\) is countably compact. Let \(C\) be a countably infinite subset of \(X\). We now assert \(C \subseteq c_{T_{\text{ACP}}} (C)\). Since \((X, T)\) is countably compact, clearly \(C \subseteq c(C)\). Since \((X, T)\) is countably AP and \(C\) is countable, we have that
\[
c(C) = [C]_{AP} = [C]_{ACP} = [C]_{ACP-T_{\text{ACP}}} = c_{T_{\text{ACP}}} (C).
\]
Thus \(C \subseteq c_{T_{\text{ACP}}} (C)\), and so \((X, T_{\text{ACP}})\) is countably compact. Therefore, \((X, T_{\text{ACP}})\) is a Fréchet-Urysohn space.

\[\square\]

Note that if \((X, T)\) is an AP (Fréchet-Urysohn) space, then \(T = T_{\text{ACP}}\) (resp. \(T = T_{\text{seq}}\)); equivalently, for each subset \(A\) of \(X\), \(c(A) = [A]_{ACP}\) (resp. \(c(A) = [A]_{seq}\)).

**Corollary 2.17.** If \((X, T)\) is a countably compact and countably AP space, then the Fréchet-Urysohn expansions \((X, T_{\text{seq}})\) and \((X, T_{\text{ACP}})\) of \((X, T)\) are homeomorphic; i.e., \(T_{\text{seq}} = T_{\text{ACP}}\).

**Proof.** By Lemmas 2.11 and 2.12(2) and Theorem 2.16, \((X, T_{\text{seq}})\) and \((X, T_{\text{ACP}})\) are countably compact and Fréchet-Urysohn spaces with \(T \subset T_{\text{seq}}\) and \(T \subset T_{\text{ACP}}\). Note that for each subset \(A\) of \(X\), \(c_{T_{\text{seq}}}(A) = [A]_{seq}\), \(c_{T_{\text{ACP}}}(A) = [A]_{ACP}\), and \([A]_{seq} \subset [A]_{ACP}\). Hence it is sufficient to prove that for each subset \(A\) of \(X\), \([A]_{ACP} \subset [A]_{seq}\). Let \(A\) be a subset of \(X\) and \(x \in [A]_{ACP}\). Then, by definition of \([\cdot]_{ACP}\), there exists a countable subset \(C\) of \(A\) such that \(\{x\} = c(C) - C\). Since \((X, T)\) is countably compact and countably AP, by Theorem 2.7, it is countably Fréchet-Urysohn. It follows that there exists a sequence \((x_n)\) of points of \(C\) such that \((x_n)\) converges to \(x\) and so \(x \in [C]_{seq}\). Thus \(x \in [A]_{seq}\). \(\square\)

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