ENDPOINT ESTIMATES FOR MULTILINEAR INTEGRAL OPERATORS

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Abstract. In this paper, the endpoint estimates for some multilinear operators related to certain integral operators are obtained. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

1. Introduction and Notations

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1-6]). In [10], the boundedness properties of the commutators for the extreme values of p are obtained. The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type integral operators. The operators include Littlewood-Paley operators and Marcinkiewicz operators.

First, let us introduce some notations (see [7-9], [14-16]). Throughout this paper, Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For a cube Q and a locally integrable function $f$, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, $f$ is said to belong to $BMO(\mathbb{R}^n)$ if $f^\# \in L^\infty$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$; We also define the central $BMO$ space by $CMO(\mathbb{R}^n)$, which is the space of those functions $f \in L_{loc}(\mathbb{R}^n)$ such that

$$||f||_{CMO} = \sup_{r > 1} ||Q(0, r)||^{-1}\int_Q |f(y) - f_Q|dy < \infty.$$ 

It is well-known that (see [8], [9])

$$||f||_{CMO} \approx \sup_{r > 1} \inf_{c \in C} ||Q(0, r)||^{-1}\int_Q |f(x) - c|dx.$$ 

Also, we give the concepts of the atom and $H^1$ space. A function $a$ is called as $H^1$ atom if there exists a cube $Q$ such that $a$ is supported on $Q$, $||a||_{L^\infty} \leq |Q|^{-1}$.
and $\int a(x)\,dx = 0$. It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [9]).

For $k \in \mathbb{Z}$, define $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by $\chi_k$ the characteristic function of $C_k$ and $\tilde{\chi}_k$ the characteristic function of $C_k$ for $k \geq 1$ and $\tilde{\chi}_0$ the characteristic function of $B_0$.

**Definition 1.** Let $0 < p < \infty$ and $\alpha \in R$.

1. The homogeneous Herz space $\dot{K}_p^\alpha(R^n)$ is defined by
   $$\dot{K}_p^\alpha(R^n) = \{f \in L^p_{\text{loc}}(R^n \setminus \{0\}) : ||f||_{\dot{K}_p^\alpha} < \infty\},$$
   where
   $$||f||_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha} ||f\chi_k||_{L^p};$$

2. The nonhomogeneous Herz space $K_p^\alpha(R^n)$ is defined by
   $$K_p^\alpha(R^n) = \{f \in L^p_{\text{loc}}(R^n) : ||f||_{K_p^\alpha} < \infty\},$$
   where
   $$||f||_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha} ||f\tilde{\chi}_k||_{L^p}.$$

If $\alpha = n(1 - 1/p)$, we denote that $\dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n)$, $K_p^\alpha(R^n) = K_p(R^n)$.

**Definition 2.** Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions $f$ on $R^n$ such that
   $$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

**Definition 3.** Let $1 < p < \infty$.

1. The homogeneous Herz type Hardy space $H\dot{K}_p(R^n)$ is defined by
   $$H\dot{K}_p(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(R^n)\},$$
   where
   $$||f||_{H\dot{K}_p} = ||G(f)||_{\dot{K}_p};$$

2. The nonhomogeneous Herz type Hardy space $HK_p(R^n)$ is defined by
   $$HK_p(R^n) = \{f \in S'(R^n) : G(f) \in K_p(R^n)\},$$
   where
   $$||f||_{HK_p} = ||G(f)||_{K_p};$$

where $G(f)$ is the grand maximal function of $f$.

The Herz type Hardy spaces have the atomic decomposition characterization.
Definition 4. Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(n(1-1/p), p)$-atom (or a central $(n(1-1/p), p)$-atom of restrict type), if

1) Suppose $C \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$);
2) $\|a\|_{L^p} \leq |B(0, r)|^{1/p-1}$;
3) $\int_{\mathbb{R}^n} a(x) dx = 0$.

Lemma 1. (see [8], [15]). Let $1 < p < \infty$. A temperate distribution $f$ belongs to $\dot{H}_{K_p}(\mathbb{R}^n)$ (or $H_{K_p}(\mathbb{R}^n)$) if and only if there exist central $(n(1-1/p), p)$-atoms (or central $(n(1-1/p), p)$-atoms of restrict type) $a_j$ supported on $B_j = B(0, 2^j)$ and constants $\lambda_j$, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=\infty}^\delta \lambda_j a_j$ (or $f = \sum_{j=0}^\delta \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, and

$$\|f\|_{\dot{H}_{K_p}}(\text{or } \|f\|_{H_{K_p}}) \sim \sum_j |\lambda_j|.$$ 

2. Theorems

In this paper, we will study a class of multilinear operators related to some non-convolution type integral operators, whose definition are following.

Let $m_l$ be the positive integers ($j = 1, \ldots, l$), $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $\mathbb{R}^n$ ($j = 1, \ldots, l$). Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\beta| \leq m_j} \frac{1}{|\beta|} \partial^\beta A_j(y)(x - y)^\beta$$

and

$$Q_{m_j+1}(A_j; x, y) = R_{m_j}(A_j; x, y) - \sum_{|\beta| = m_j} \frac{1}{|\beta|!} D^\beta A_j(x)(x - y)^\beta.$$ 

Fixed $0 \leq \delta < n$. Let $F_t(x, y)$ define on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x, y)f(y)dy$$

and

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, y) |x - y|^\alpha F_t(x, y)f(y)dy$$

for every bounded and compactly supported function $f$. Let $H$ be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to $H$. Then, the multilinear operator related to $F_t$ is defined by

$$T_t^A(f)(x) = ||F_t^A(f)(x)||,$$

where $F_t$ satisfies: for fixed $\varepsilon > 0$,

$$||F_t(x, y)|| \leq C|x - y|^{-n+\delta}$$

and

$$||F_t(y, x) - F_t(z, x)|| \leq C|y - z|\varepsilon|x - z|^{-n-\varepsilon+\delta}.$$
Let Theorem 4 is also hold for nonhomogeneous Herz and Herz type Hardy properties of the multilinear operators operator are obtained. In this paper, we will study the endpoint continuity 

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has been widely studied by many authors (see [3-6]). In [2], the weak (\( H^1, L^1 \))-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the endpoint continuity properties of the multilinear operators \( T_3^A \) and \( \tilde{T}_3^A \). In Section 4, we will give some applications of Theorems in this paper.

Now we state our results as following.

**Theorem 1.** Let \( 0 \leq \delta < n \) and \( D^\beta A_j \in BMO(R^n) \) for all \( \beta \) with \( |\beta| = m_j \) and \( j = 1, \ldots, l \). Suppose that \( T_3 \) is bounded from \( L^r(R^n) \) to \( L^s(R^n) \) for any \( r, s \in (1, +\infty] \) with \( 1 < r < n/\delta \) and \( 1/s = 1/r - \delta/n \). Then \( T_3^A \) is bounded from \( L^{n/\delta}(R^n) \) to \( BMO(R^n) \).

**Theorem 2.** Let \( 0 \leq \delta < n \) and \( D^\beta A_j \in BMO(R^n) \) for all \( \beta \) with \( |\beta| = m_j \) and \( j = 1, \ldots, l \). Suppose that \( \tilde{T}_3^A \) is bounded from \( L^r(R^n) \) to \( L^s(R^n) \) for any \( r, s \in (1, +\infty] \) with \( 1 < r < n/\delta \) and \( 1/s = 1/r - \delta/n \). Then \( \tilde{T}_3^A \) is bounded from \( H^1(R^n) \) to \( L^{n/(n-\delta)}(R^n) \).

**Theorem 3.** Let \( 0 \leq \delta < n \), \( 1 < p < n/\delta \) and \( D^\beta A_j \in BMO(R^n) \) for all \( \beta \) with \( |\beta| = m_j \) and \( j = 1, \ldots, l \). Suppose that \( T_3 \) is bounded from \( L^r(R^n) \) to \( L^s(R^n) \) for any \( r, s \in (1, +\infty] \) with \( 1 < r < n/\delta \) and \( 1/s = 1/r - \delta/n \). Then \( T_3^A \) is bounded from \( B_2^\alpha(R^n) \) to \( CMO(R^n) \).

**Theorem 4.** Let \( 0 \leq \delta < n \), \( 1 < p < n/\delta \), \( 1/q = 1/p - \delta/n \) and \( D^\beta A_j \in BMO(R^n) \) for all \( \beta \) with \( |\beta| = m_j \) and \( j = 1, \ldots, l \). Suppose that \( \tilde{T}_3^A \) is bounded from \( L^r(R^n) \) to \( L^s(R^n) \) for any \( r, s \in (1, +\infty] \) with \( 1 < r < n/\delta \) and \( 1/s = 1/r - \delta/n \). Then \( \tilde{T}_3^A \) is bounded from \( HK_p(R^n) \) to \( K_\alpha^p(R^n) \) with \( \alpha = n(1 - 1/p) \).

**Remark.** Theorem 4 is also hold for nonhomogeneous Herz and Herz type Hardy space.

### 3. Proofs of Theorems

To prove the theorems, we need the following lemma.
Lemma 2. (see [6]) Let $A$ be a function on $\mathbb{R}^n$ and $D^\beta A \in L^q(\mathbb{R}^n)$ for $|\beta| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\beta| = m} \left( \frac{1}{Q(x, y)} \int_{Q(x, y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $Q(x, y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

Proof of Theorem 1. It is only to prove that there exists a constant $C_Q$ such that

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_Qdx| \leq C||f||_{L^{n/q}}$$

holds for any cube $Q$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum |\beta| = m \tilde{A}_jQ^\beta x^\beta$.

then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)Q^\beta$ for $|\beta| = m_j$.

We write, for $f_1 = f\chi_Q$ and $f_2 = f\chi_{R^c Q}$,

$$F^A_1(f)(x) = \int_{R^n} \prod_{j=1}^2 R_{m_{j+1}}(\tilde{A}_j; x, y) f_t(x, y) f(y) dy$$

$$= \int_{R^n} \prod_{j=1}^2 R_{m_{j+1}}(\tilde{A}_j; x, y) f_t(x, y) f_2(y) dy$$

$$+ \int_{R^n} \prod_{j=1}^2 R_{m_1} (\tilde{A}_j; x, y) f_t(x, y) f_1(y) dy$$

$$- \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{R^n} R_{m_2} (\tilde{A}_2; x, y) (x - y)^{\beta_1}$$

$$\times D^\beta \tilde{A}_2(x, y) f_t(x, y) f_1(y) dy$$

$$- \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{R^n} R_{m_1} (\tilde{A}_1; x, y) (x - y)^{\beta_2}$$

$$\times D^\beta \tilde{A}_1(x, y) f_t(x, y) f_1(y) dy$$

$$+ \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!} \int_{R^n} (x - y)^{\beta_1 + \beta_2} D^\beta_1 \tilde{A}_1(x) D^\beta_2 \tilde{A}_2(x)$$

$$\times f_t(x, y) f_1(y) dy.$$

Then

$$\left| T^A(f)(x) - T^A(f_2)(x_0) \right|$$

$$= \left| ||F^A_1(f)(x)|| - ||F^A_1(f_2)(x_0)|| \right|$$
thus, by \( \| F_t^A(f)(x) - F_t^A(f_2)(x_0) \| \)
\[
\leq \left\| \prod_{j=1}^{2} R_m(\tilde{A}_j; x, y) \frac{F_t(x, y) f_1(y)}{|x - y|^m} \right\|
\]
\[
+ \left\| \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{\mathbb{R}^n} R_m(\tilde{A}_1; x, y)(x - y)^{\beta_1} D^{\beta_1} \tilde{A}_1(y) F_t(x, y, f_1(y)) dy \right\|
\]
\[
+ \left\| \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{\mathbb{R}^n} R_m(\tilde{A}_2; x, y)(x - y)^{\beta_2} D^{\beta_2} \tilde{A}_2(y) F_t(x, y, f_1(y)) dy \right\|
\]
\[
+ \left\| \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{\beta_1! \beta_2!} \int_{\mathbb{R}^n} (x - y)^{\beta_1 + \beta_2} D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) F_t(x, y, f_1(y)) dy \right\|
\]
\[
+ |T^A_3(f_2)(x) - T^A_3(f_2)(x_0)|
\]
\[
:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\]

thus,
\[
\frac{1}{|Q|} \int_Q |T^A_3(f)(x) - T^A_3(f_2)(x_0)| dx
\]
\[
\leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx
\]
\[
+ \frac{1}{|Q|} \int_Q I_4(x) dx + \frac{1}{|Q|} \int_Q I_5(x) dx
\]
\[
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Now, let us estimate \( I_1, I_2, I_3, I_4 \) and \( I_5 \), respectively. For \( I_1 \), by Lemma 2, we get, for \( x \in Q \) and \( y \in Q \),
\[
R_m(\tilde{A}_j; x, y) \leq C|x - y|^m \sum_{|\alpha_j| = m_j} \| D^{\alpha_j} A_j \|_{BMO},
\]

thus, by \( (L^{n/\delta}, L^\infty) \)-boundedness of \( T_3 \), we get
\[
I_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \| D^{\alpha_j} A_j \|_{BMO} \right) \frac{1}{|Q|} \int_Q |T_3(f_1)(x)| dx
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j| = m_j} \| D^{\alpha_j} A_j \|_{BMO} \right) \| T_3(f_1) \|_{L^\infty}.
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{\text{BMO}} \right) \|f\|_{L^{n/\delta}};
\]

For \( I_2 \), by \((L^p, L^q)\)-boundedness of \( T_\delta \) for \( 1/q = 1/p - \delta/n \), \( n/\delta > p > 1 \) and Hölder inequality, we get

\[
I_2 \leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{\text{BMO}} \sum_{|\beta_1|=m_1} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 f_1)(x)| \, dx
\]

\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{\text{BMO}} \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_{R^n} |D^{\beta_1} \tilde{A}_1 f_1(x)|^q \, dx \right)^{1/q}
\]

\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{\text{BMO}} \times \sum_{|\beta_1|=m_1} |Q|^{-1/q} \left( \int_{R^n} |D^{\beta_1} \tilde{A}_1 f_1(x)|^p \, dx \right)^{1/p}
\]

\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{\text{BMO}} \times |Q|^{-1/q} \left( \int_{R^n} |D^{\beta_1} A_1(x) - (D^{\beta_1} A_1)_Q|^q \, dx \right)^{1/q} \|f\|_{L^{n/\delta}}
\]

\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{\text{BMO}} \right) \|f\|_{L^{n/\delta}};
\]

For \( I_3 \), similar to the proof of \( I_2 \), we get

\[
I_3 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{\text{BMO}} \right) \|f\|_{L^{n/\delta}};
\]

Similarly, for \( I_4 \), choose \( 1 < p < n/\delta \) and \( q, r_1, r_2 > 1 \) such that \( 1/q = 1/p - \delta/n \) and \( 1/r_1 + 1/r_2 + p\delta/n = 1 \), we obtain, by Hölder inequality,

\[
I_4 \leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)| \, dx
\]

\[
\leq C \sum_{|\beta_1|=m_1, |\beta_2|=m_2} \left( \frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)|^q \, dx \right)^{1/q}
\]
\[
\begin{align*}
\leq & \; C \sum_{|\beta| = m_1, |\beta_2| = m_2} |Q|^{-1/\alpha} \left( \int_{R^n} |D^{\beta_1} \tilde{A}_1(x) D^{\beta_2} \tilde{A}_2(x) f_1(x)|^p \, dx \right)^{1/p} \\
\leq & \; C \sum_{|\beta| = m_1, |\beta_2| = m_2} \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_1} \tilde{A}_1(x)|^{p_{r_1}} \, dx \right)^{1/p_{r_1}} \\
\times & \left( \frac{1}{|Q|} \int_{Q} |D^{\beta_2} \tilde{A}_2(x)|^{p_{r_2}} \, dx \right)^{1/p_{r_2}} \|f\|_{L^{n/s}} \\
\leq & \; C \sum_{j=1}^{2} \left( \sum_{|\beta| = m_j} \|D^{\beta} A_j\|_{BMO} \right) \|f\|_{L^{n/s}} \\
\end{align*}
\]

For \( I_5 \), we write
\[
F^\#_1(f_2)(x) - F^\#_1(f_2)(x_0) \\
= \int_{R^n} \left( \frac{F_1(x, y)}{|x-y|^m} - \frac{F_1(x_0, y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) f_2(y) dy \\
+ \int_{R^n} \left[ R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \cdot \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x-y|^m} \right] F_1(x_0, y) f_2(y) dy \\
+ \int_{R^n} \left[ R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \cdot \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x-y|^m} \right] F_1(x, y) f_2(y) dy \\
- \sum_{|\beta| = m_1} \frac{1}{\beta_1!} \int_{R^n} \left[ R_{m_2}(\tilde{A}_2; x, y) \cdot \frac{(x-y)^{\beta_1}}{|x-y|^m} \right] D^{\beta_1} \tilde{A}_1(y) f_2(y) dy \\
- \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{R^n} \left[ R_{m_1}(\tilde{A}_1; x, y) \cdot \frac{(x-y)^{\beta_2}}{|x-y|^m} \right] D^{\beta_2} \tilde{A}_2(y) f_2(y) dy \\
+ \sum_{|\beta| = m_1, |\beta_2| = m_2} \frac{1}{\beta_1! \beta_2!} \int_{R^n} \left[ \frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m} \right] F_1(x, y) \\
- \frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m} \left[ F_1(x_0, y) \right] D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) f_2(y) dy \\
= I^{(1)}_5 + I^{(2)}_5 + I^{(3)}_5 + I^{(4)}_5 + I^{(5)}_5 + I^{(6)}_5.
\]

By Lemma 1 and the following inequality (see [16])
\[
|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,
\]
we know that, for $x \in Q$ and $y \in 2^{k+1} \hat{Q} \setminus 2^k \hat{Q}$,

$$|R_{m_j}(\tilde{A}_j; x, y)| \leq C|x-y|^{m_j} \sum_{|\beta|=m_j} (||D^\beta A_j||_{BMO}$$

$$+ ||(D^\beta A_j)_{\tilde{Q}(x,y)} - (D^\beta A_j)_{\hat{Q}}||)$$

$$\leq Ck|x-y|^{m_j} \sum_{|\beta|=m_j} ||D^\beta A_j||_{BMO}.$$  

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \hat{Q}$, we obtain, by the condition of $F_t$,

$$||f^{(1)}|| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^{\infty}}{|x_0 - y|^{n+\varepsilon - \delta}} \right)$$

$$\times \prod_{j=1}^{2} ||R_{m_j}(\tilde{A}_j; x, y)||f_2(y)||dy$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right)$$

$$\times \sum_{k=0}^{\infty} \int_{2^{k+1} \hat{Q} \setminus 2^k \hat{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^{\infty}}{|x_0 - y|^{n+\varepsilon - \delta}} \right) ||f(y)||dy$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\varepsilon k}) ||f||_{L^{n/\delta}}$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}.$$  

For $I_j^{(2)}$, by the formula (see [6]):

$$R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y) = \sum_{|\gamma|<m} \frac{1}{|\gamma|!} R_{m-|\gamma|}(D^\gamma \tilde{A}_j; x, x_0)(x - y)^\gamma$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y)|$$

$$\leq C \sum_{|\gamma|<m_j} \sum_{|\beta|=m_j} |x - x_0|^{m_j-|\gamma|} |x - y|^{|\gamma|} ||D^\beta A_j||_{BMO}.$$
thus

\[ ||I_5^{(2)}|| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) \times \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} k^{\frac{1}{2k+1}} |f(y)| dy \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{L^{n/\delta}} \]

Similarly,

\[ ||I_5^{(3)}|| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{L^{n/\delta}} \]

For \( I_5^{(4)} \), taking \( r > 1 \) such that \( 1/r + \delta/n = 1 \), then

\[ ||I_5^{(4)}|| \leq C \sum_{|\beta_j|=m_j} \int_{R^n} \left| \frac{(x-y)^{\beta_1}F_1(x,y) - (x_0-y)^{\beta_1}F_1(x,y)}{|x-y|^m} \right| \]

\[ \times |R_{m_2}(\tilde{A}_2;x,y)||D^{\beta_1}\tilde{A}_1(y)||f_2(y)| dy | R_{m_2}(\tilde{A}_2;x_0,y) | \]

\[ \times \int_{R^n} \left| \frac{(x-y)^{\beta_2}F_2(x_0,y)}{|x_0-y|^m} \right||D^{\beta_2}\tilde{A}_1(y)||f_2(y)| dy \]

\[ \leq C \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\epsilon k}) \]

\[ \times \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_1}\tilde{A}_1(y)|^r dy \right)^{1/r} ||f||_{L^{n/\delta}} \]

\[ \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{L^{n/\delta}} \]

Similarly,

\[ ||I_5^{(5)}|| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{L^{n/\delta}} \]
For $I_5^{(6)}$, taking $r_1, r_2 > 1$ such that $\delta/n + 1/r_1 + 1/r_2 = 1$, then

$$
||I_5^{(6)}|| 
\leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \int_{\mathbb{R}^n} \left\| \frac{(x-y)^{\delta_1 + \delta_2} F_i(x,y)}{|x-y|^m} \right\|_{L^\infty} \bigg( \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \left( 2^{-k} + 2^{-\varepsilon k} \right) ||f||_{L^{n/\delta}} \bigg)

\times \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\delta_1} \tilde{A}_1(y)|^{r_1} dy \right)^{1/r_1} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\delta_2} \tilde{A}_2(y)|^{r_2} dy \right)^{1/r_2}

\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} ||D^{\delta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}.

Thus

$$
I_5 \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j| = m_j} ||D^{\delta_j} A_j||_{BMO} \right) ||f||_{L^{n/\delta}}.
$$

This completes the proof of Theorem 1.

**Proof of Theorem 2.** It is only to show that there exists a constant $C > 0$ such that for every $H^1$-atom $a$ (that is that $a$ satisfies: supp $a \subset Q = Q(x_0, d)$, $||a||_{L^\infty} \leq |Q|^{-1}$ and $\int a(y)dy = 0$ (see [9])), the following holds:

$$
||\tilde{T}_\delta^A(a)||_{L^{n/(n-\delta)}} \leq C.
$$

Without loss of generality, we may assume $l = 2$. Write

$$
\int_{\mathbb{R}^n} \left[ \tilde{T}_\delta^A(a)(x) \right]^{n/(n-\delta)} dx
\leq \int_{|x-x_0| \leq 2d} + \int_{|x-x_0| > 2d} \left[ \tilde{T}_\delta^A(a)(x) \right]^{n/(n-\delta)} dx := J_1 + J_2.
$$

For $J_1$, by the $(L^p, L^q)$-boundedness of $\tilde{T}_\delta^A$ for $1/q = 1/p - \delta/n$, $n/\delta > p > 1$, we get

$$
J_1 \leq C ||\tilde{T}_\delta^A(a)||_{L^q}^{n/(n-\delta)q} ||2Q|^{1-n/((n-\delta)q)}
\leq C ||a||_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \leq C.
$$

To obtain the estimate of $J_2$, we denote that

$$
\tilde{A}_j(x) = A_j(x) - \sum_{|\beta_j| = m_j} \frac{1}{|Q|} (D^{\delta_j} A_j)_{2Qx^3}.
$$
This completes the proof of Theorem 2. We write, by the vanishing moment of \( a \),

\[
\begin{align*}
\mathcal{F}_t^4(a)(x) &= \int_{\mathbb{R}^n} \left[ \frac{F_t(x, y)}{|x - y|^m} - \frac{F_t(x, x_0)}{|x - x_0|^m} \right] R_{m_1}(\tilde{A}; x, y) R_{m_2}(\tilde{A}; x, y) a(y) dy \\
+ \int_{\mathbb{R}^n} \frac{F_t(x, x_0)}{|x - x_0|^m} \left[ R_{m_1}(\tilde{A}; x, y) R_{m_2}(\tilde{A}; x, y) - R_{m_1}(\tilde{A}_1; x, x_0) R_{m_2}(\tilde{A}_2; x, x_0) \right] a(y) dy \\
- \sum_{|k| = m_2 \leq m} \int_{\mathbb{R}^n} \left[ \frac{F_t(x, y)(x - y)^{\beta_2}}{|x - y|^m} - \frac{F_t(x, x_0)(x - x_0)^{\beta_2}}{|x - x_0|^m} \right] R_{m_1}(\tilde{A}_1; x, y) a(y) dy \\
\times D^{\beta_2} \tilde{A}_2(x) a(y) dy \\
- \sum_{|k| = m_1 \leq m} \int_{\mathbb{R}^n} \left[ \frac{F_t(x, y)(x - y)^{\beta_1}}{|x - y|^m} - \frac{F_t(x, x_0)(x - x_0)^{\beta_1}}{|x - x_0|^m} \right] R_{m_2}(\tilde{A}_2; x, y) a(y) dy \\
\times D^{\beta_1} \tilde{A}_1(x) a(y) dy \\
+ \sum_{|k| = m_1, |k| = m_2 \leq m} \int_{\mathbb{R}^n} \left[ \frac{F_t(x, y)(x - y)^{\beta_1 + \beta_2}}{|x - y|^m} - \frac{K(x, x_0)(x - x_0)^{\beta_1 + \beta_2}}{|x - x_0|^m} \right] D^{\beta_1} \tilde{A}_1(x) D^{\beta_2} \tilde{A}_2(x) a(y) dy,
\end{align*}
\]

similar to the proof of Theorem 1, we obtain

\[
J_2 \leq C \left[ \prod_{j=1}^2 \left( \sum_{|k| = m_j} \|D^k A_j\|_{BMO} \right) \right]^{n/(n-\delta)} \\
\times \sum_{k=1}^{\infty} k^2 \left[ 2^{-kn/(n-\delta)} + 2^{-kn/\delta} \right] \\
\leq C.
\]

This completes the proof of Theorem 2. \( \square \)

**Proof of Theorem 3.** It suffices to prove that there exists a constant \( C_Q \) such that

\[
\frac{1}{|Q|} \int_Q |T^4_{B^\delta}(f)(x) - C_Q| dx \leq C ||f||_{B^\delta}
\]
holds for any cube $Q = Q(0, d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\beta| = m} \frac{1}{m!}(D^\beta A_j)_Q x^\beta$, then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\beta \tilde{A}_j = D^\beta A_j - (D^\beta A_j)_Q$ for $|\beta| = m_j$. We write, for $f_1 = f \chi_Q$ and $f_2 = f \chi_{\tilde{Q}}$, 

$$F^4_2(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{A}_j; x, y)}{|x - y|^m} F_1(x, y) f(y)\,dy$$

$$= \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j+1}(\tilde{A}_j; x, y)}{|x - y|^m} F_1(x, y) f_2(y)\,dy$$

$$+ \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} F_1(x, y) f_1(y)\,dy$$

$$- \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_m(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) F_1(x, y) f_1(y)\,dy$$

$$- \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{R_m(\tilde{A}_1; x, y)(x - y)^{\beta_2}}{|x - y|^m} D^{\beta_2} \tilde{A}_2(y) F_1(x, y) f_1(y)\,dy$$

$$+ \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{\beta_1! \beta_2!} \int_{R^n} \frac{(x - y)^{\beta_1 + \beta_2}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) F_1(x, y) f_1(y)\,dy,$$

then

$$\frac{1}{|Q|} \int_Q \left| T_{x}^4(f)(x) - T_{x}^2(f_2)(0) \right| dx$$

$$\leq \frac{1}{|Q|} \int_Q \left\| \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)}{|x - y|^m} F_1(x, y) f_1(y)\,dy \right\| dx$$

$$+ \frac{1}{|Q|} \int_Q \left\| \sum_{|\beta_1| = m_1} \frac{1}{\beta_1!} \int_{R^n} \frac{R_m(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) F_1(x, y) f_1(y)\,dy \right\| dx$$

$$+ \frac{1}{|Q|} \int_Q \left\| \sum_{|\beta_2| = m_2} \frac{1}{\beta_2!} \int_{R^n} \frac{R_m(\tilde{A}_1; x, y)(x - y)^{\beta_2}}{|x - y|^m} D^{\beta_2} \tilde{A}_2(y) F_1(x, y) f_1(y)\,dy \right\| dx$$

$$+ \frac{1}{|Q|} \int_Q \left\| \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{\beta_1! \beta_2!} \int_{R^n} \frac{(x - y)^{\beta_1 + \beta_2}}{|x - y|^m} D^{\beta_1} \tilde{A}_1(y) D^{\beta_2} \tilde{A}_2(y) F_1(x, y) f_1(y)\,dy \right\|.$$
Similar to the proof of Theorem 1, we get, for $1/s = 1/r - \delta/n$, $1 < r < p$, $1 < t_1, t_2 < \infty$ and $1/t_1 + 1/t_2 + r/p = 1$,

\[
L_1 \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |T_\delta(f_1)(x)| dx
\]
\[
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \left( \frac{1}{|Q|} \int_Q |T_\delta(f_1)(x)|^q dx \right)^{1/q}
\]
\[
\leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) d^{-n(1/p-\delta/n)} \|\chi_Q\|_{L^p} \|f\|_{B^s_p};
\]

\[
L_2 \leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \sum_{|\beta_1|=m_1} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 f_1)(x)| dx
\]
\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\beta_1} \tilde{A}_1 f_1)(x)|^s dx \right)^{1/s}
\]
\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \sum_{|\beta_1|=m_1} |Q|^{-1/s} \|D^{\beta_1} A_1 - (D^{\beta_1} A_1)_Q\|_{L^r} f_1\|_{L^p}
\]
\[
\leq C \sum_{|\beta_2|=m_2} \|D^{\beta_2} A_2\|_{BMO} \times \sum_{|\beta_1|=m_1} \left( \frac{1}{|Q|} \int_Q |D^{\beta_1} \tilde{A}_1(y)|^{pr/(p-r)} dy \right)^{(p-r)/pr} |Q|^{\delta/n-1/p} \|f_1\|_{L^p}.
\]
\[
\begin{align*}
L_4 & \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)|dx \\
& \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} \left( \frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\beta_1} \tilde{A}_1 D^{\beta_2} \tilde{A}_2 f_1)(x)|^s dx \right)^{1/s} \\
& \leq C \sum_{|\beta_1| = m_1, |\beta_2| = m_2} |Q|^{-1/s} \left( \int_{R^n} |D^{\beta_1} \tilde{A}_1(x) D^{\beta_2} \tilde{A}_2(x) f_1(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{|\beta_1| = m_1} \left( \frac{1}{|Q|} \int_Q |D^{\beta_1} \tilde{A}_1(x)|^{r_{t_1}} dx \right)^{1/r_{t_1}} \\
& \times \sum_{|\beta_2| = m_2} \left( \frac{1}{|Q|} \int_Q |D^{\beta_2} \tilde{A}_2(x)|^{r_{t_2}} dx \right)^{1/r_{t_2}} |Q|^{\delta/n - 1/p} \|
 f_1 \|_{L^p} \\
& \leq C \sum_{j=1}^2 \left( \sum_{|\beta_j| = m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^p;}
\end{align*}
\]

For \( L_5 \), we write, for \( x \in Q \),
\[
F^\tilde{A}_i(f_2)(x) - F^\tilde{A}_i(f_2)(0)
\]
\[
= \int_{R^n} \left( \frac{F_i(x, y)}{|x - y|^m} - \frac{F_i(0, y)}{|y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y) f_2(y) dy \\
+ \int_{R^n} \left( R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; 0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|y|^m} F_i(0, y) f_2(y) dy \\
+ \int_{R^n} \left( R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; 0, y) \right) \frac{R_{m_1}(\tilde{A}_1; 0, y)}{|y|^m} F_i(0, y) f_2(y) dy \\
- \sum_{|\beta_1| = m_1} \frac{1}{|\beta_1|!} \int_{R^n} \left[ \frac{R_{m_2}(\tilde{A}_2; x, y)(x - y)^{\beta_1}}{|x - y|^m} F_i(x, y) \right]
\]
\[- \frac{R_{m_2}(\hat{A}_2; 0, y)(-y)^{\beta_1}F_1(0, y)}{|y|^m} D^{\beta_1}\hat{A}_1(y)f_2(y)dy \]
\[- \sum_{|\beta_1|=m_2} \frac{1}{\beta_1!} \int_{R^n} \left[ \frac{R_{m_1}(\hat{A}_1; x, y)(x-y)^{\beta_2}}{|x-y|^m} F_1(x, y) \right] \]
\[- \frac{R_{m_1}(\hat{A}_1; 0, y)(-y)^{\beta_2}F_1(0, y)}{|y|^m} \]
\[\sum_{|\beta_1|=m_1, |\beta_2|=m_2} \frac{1}{\beta_1!\beta_2!} \int_{R^n} \left[ \frac{(x-y)^{\beta_1+\beta_2}}{|x-y|^m} F_1(x, y) - \frac{(-y)^{\beta_1+\beta_2}}{|y|^m} F_1(0, y) \right] \]
\[D^{\beta_2}\hat{A}_2(y)f_2(y)dy \]
\[= L_5^{(1)} + L_5^{(2)} + L_5^{(3)} + L_5^{(4)} + L_5^{(5)} + L_5^{(6)} \]

Similar to the proof of Theorem 1, we get, for $1 < r_1, r_2 < \infty$ and $1/r_1 + 1/r_2 + 1/p = 1$,

\[\|L_5^{(1)}\| \leq C \int_{R^n} \left( \frac{|x|}{|y|^{m+n+1-\delta}} + \frac{|x|^\varepsilon}{|y|^{m+n+r-\delta}} \right) \prod_{j=1}^{2} |R_{m_j}(\hat{A}_j; x, y)||f_2(y)|dy \]
\[\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) \]
\[\times \sum_{k=0}^{\infty} \int_{2^{k+1}2^{k}\vartriangle 2^{k}} k^2 \left( \frac{|x|}{|y|^{n+1-\delta}} + \frac{|x|^\varepsilon}{|y|^{n+r-\delta}} \right) |f(y)|dy \]
\[\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) \]
\[\times \sum_{k=0}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) (2^k, d)^{n(1/p-\delta/n)} ||f||_{L^p(A_2\vartriangle A_1)} \]
\[\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) ||f||_{B^{2}_{p}} \]
\[\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{B^{2}_{p}} ;
\]
\[\|L_5^{(2)}\| \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}2^{k}\vartriangle 2^{k}} k \frac{|x|}{|y|^{n+1-\delta}} |f(y)|dy \]
\[
\begin{align*}
\|L^3_N\| & \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{R^+_p} \\
\|L^4_N\| & \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^+_p} \\
\|L^5_N\| & \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \int_{R^n} \left| \frac{(x-y)^{\beta_j} F_k(x,y)}{|x-y|^m} - \frac{(-y)^{\beta_j} F_k(0,y)}{|y|^m} \right| \right) \\
& \quad \times |R_{m_2}(\tilde{A}_2;x,y)| |D^{\beta_j} \tilde{A}_1(y)||f_2(y)|dy \\
& \quad + C \sum_{|\beta_2|=m_2} \left| D^{\beta_2} A_2 \right|_{BMO} \\
& \quad \times \sum_{k=1}^\infty \left( 2^{-k} + 2^{-\varepsilon k} \right) \frac{|D^{\beta_2} A_2\|_{BMO}|f_2(y)|}{|y|^m} \right) \\
& \quad \times \frac{1}{|2^k Q|} \int_{2^k Q} |D^{\beta_1} \tilde{A}_1(y)|^{p'} dy \\
& \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \|D^{\beta_j} A_j\|_{BMO} \right) \|f\|_{B^+_p} \\
\|L^6_N\| & \leq C \prod_{j=1}^2 \left( \sum_{|\beta_j|=m_j} \int_{R^n} \left| \frac{(x-y)^{\beta_j+\beta_2} F_k(x,y)}{|x-y|^m} - \frac{(-y)^{\alpha_1+\alpha_2} F_k(0,y)}{|y|^m} \right| \right) \\
& \quad \times |D^{\beta_1} \tilde{A}_1(y)||D^{\beta_2} \tilde{A}_2(y)||f_2(y)|dy \\
& \leq C \sum_{k=1}^\infty \left( 2^{-k} + 2^{-\varepsilon k} \right) \frac{|D^{\beta_2} A_2\|_{BMO}|f_2(y)|}{|y|^m} \right) \\
& \quad \times \sum_{|\beta_1|=m_1} \left( 2^{-\varepsilon k} \right) \frac{|D^{\beta_1} \tilde{A}_1(y)|^{r_1} dy}{|y|^m} \\
& \quad \times \sum_{|\beta_2|=m_2} \left( 2^{-\varepsilon k} \right) \frac{|D^{\beta_2} \tilde{A}_2(y)|^{r_2} dy}{|y|^m} \\
\end{align*}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{B^*_p}.
\]
Thus
\[
L_5 \leq C \prod_{j=1}^{2} \left( \sum_{|\beta_j|=m_j} ||D^{\beta_j}A_j||_{BMO} \right) ||f||_{B^*_p}.
\]
This finishes the proof of Theorem 3.

Proof of Theorem 4. Without loss of generality, we may assume \( l = 2 \). Let \( f \in H\dot{K}_p(R^n) \), by Lemma 1, \( f = \sum_{j=-\infty}^{\infty} \lambda_j a_j \), where \( a_j \) are the central \((n(1 - 1/p), p)\)-atom with \( \text{supp} a_j \subset B_j = B(0, 2^j) \) and \( ||f||_{H\dot{K}_p} \sim \sum_j |\lambda_j| \). We write
\[
||\tilde{T}^A_{\delta}(f)||_{K_q^p} = \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} ||\chi_k \tilde{T}^A_{\delta}(f)||_{L^q}
\]
\[
\leq \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=-\infty}^{k-1} |\lambda_j| ||\chi_k \tilde{T}^A_{\delta}(a_j)||_{L^q}
\]
\[
+ \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| ||\chi_k \tilde{T}^A_{\delta}(a_j)||_{L^q} = M_1 + M_2.
\]
For \( M_2 \), by the \((L^p, L^q)\)-boundedness of \( \tilde{T}^A_{\delta} \) for \( 1/q = 1/p - \delta/n \), we get
\[
M_2 \leq C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| ||a_j||_{L^p}
\]
\[
\leq C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| 2^{jn(1/p-1)}
\]
\[
\leq C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} 2^{(k-j)n(1-1/p)}
\]
\[
\leq C \sum_{j=-\infty}^{\infty} |\lambda_j| \leq C ||f||_{H\dot{K}_p}.
\]
To obtain the estimate of \( M_1 \), we denote that
\[
\hat{A}_j(x) = A_j(x) - \sum_{|\beta_j|=m_j} \frac{1}{|\beta_j|!} (D^{\beta_j}A_j)_{2Qx^\beta}.
\]
Then \( Q_{m_1}(A_j; x, y) = Q_{m_1}(\hat{A}_j; x, y) \). We write, by the vanishing moment of \( a \),
\[
\hat{F}^A_t(a)(x)
\]
\[
= \int_{R^n} \left[ \frac{F_t(x, y)}{|x-y|^m} - \frac{F_t(x, 0)}{|x|^m} \right] R_{m_1}(\hat{A}_1; x, y) R_{m_2}(\hat{A}_2; x, y) a(y) dy
\]
+ \int_{\mathbb{R}^n} \frac{F_i(x,0)}{|x|^m} |R_{m_1}(\hat{A}_1;x,y)R_{m_2}(\hat{A}_2;x,y)\]
\quad - R_{m_1}(\hat{A}_1;x,0)R_{m_2}(\hat{A}_2;x,0)|a(y)|dy
\quad \times R_{m_1}(\hat{A}_1;x,y)D^{\beta_2}\hat{A}_2(x)a(y)dy
\quad - \sum_{|j|_1 = m_1} \int_{\mathbb{R}^n} \frac{F_i(x,0)x^{\beta_1}}{|x|^m} |R_{m_2}(\hat{A}_2;x,y) - R_{m_2}(\hat{A}_2;x,0)|
\quad \times D^{\beta_1}\hat{A}_1(x)a(y)dy
\quad + \sum_{|j|_1 = m_1, |j|_2 = m_2} \int_{\mathbb{R}^n} \frac{F_i(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m} - \frac{F_i(x,0)x^{\beta_1+\beta_2}}{|x|^m}
\quad \times D^{\beta_1}\hat{A}_1(x)D^{\beta_2}\hat{A}_2(x)a(y)dy.

Similar to the proof of Theorem 1 and Theorem 2, we get

\[ M_1 \leq C \prod_{j=1}^2 \left( \sum_{|j|_j = m_j} \||D^{\beta_j}A_j||_{BMO} \right) \]
\times \sum_{k=-\infty}^{\infty} 2^{kn(1-\delta/n)} \sum_{j=-\infty}^{k-1} |\lambda_j| \left[ \frac{2^j}{2k(n+\epsilon-\delta)} + \frac{2^{j\epsilon}}{2k(n+\epsilon-\delta)} \right] \]
\leq C \prod_{j=1}^2 \left( \sum_{|j|_j = m_j} \||D^{\beta_j}A_j||_{BMO} \right) \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} [2^{j-k} + 2^{j-k}] \]
\leq C \prod_{j=1}^2 \left( \sum_{|j|_j = m_j} \||D^{\beta_j}A_j||_{BMO} \right) \sum_{j=-\infty}^{\infty} |\lambda_j| \]
\leq C \prod_{j=1}^2 \left( \sum_{|j|_j = m_j} \||D^{\beta_j}A_j||_{BMO} \right) \||f||_{L^p}.

This completes the proof of Theorem 4. \qed
4. Applications

Now we shall apply the theorems of the paper to some particular operators such as Littlewood-Paley operators and Marcinkiewicz operators.

**Application 1.** Littlewood-Paley operator.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$ and $\mu > (3n + 2 - 2\delta)/n$. Let $\psi$ be a fixed function which satisfies the following properties:

1. $\int_{\mathbb{R}^n} \psi(x)dx = 0$,
2. $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
3. $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote that $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear operators are defined by

$$g^A_\psi(f)(x) = \left(\int_0^\infty \left|F_t^A(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$S^A_\psi(f)(x) = \left[\int_{\Gamma(x)} \left|F_t^A(f)(x,y)\right|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

and

$$g^A_\mu(f)(x) = \left[\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x - y|}\right)^{\frac{n}{2}+\mu} \left|F_t^A(f)(x,y)\right|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{\mathbb{R}^n} \prod_{j=1}^l \frac{R_{m_j+1}(A_j; x, y)}{|x - y|^{m_j}} \psi_t(x - y) f(y) dy,$$

and $\psi_t(x) = t^{-n+\delta} \psi(x/t)$ for $t > 0$. The variants of $g^A_\psi$, $S^A_\psi$ and $g^A_\mu$ are defined by

$$\tilde{g}^A_\psi(f)(x) = \left(\int_0^\infty \left|\tilde{F}_t^A(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

$$\tilde{S}^A_\psi(f)(x) = \left[\int_{\Gamma(x)} \left|\tilde{F}_t^A(f)(x,y)\right|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

and

$$\tilde{g}^A_\mu(f)(x) = \left[\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x - y|}\right)^{\frac{n}{2}+\mu} \left|\tilde{F}_t^A(f)(x,y)\right|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^l \frac{Q_{m_j+1}(A_j; x, y)}{|x - y|^{m_j}} \psi_t(x - y) f(y) dy.$$
and
\[ F^A_t(f)(x, y) = \int_{\mathbb{R}^n} \prod_{j=1}^{\gamma} Q_{m_j+1}(A_j; x, z) \frac{|x-z|^\alpha \psi_t(y-z)}{|x-z|^m} f(z) dz. \]

Set \( F_t(f)(y) = f * \psi_t(y) \). We also define that
\[ g_\psi(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}, \]
\[ S_\psi(f)(x) = \left( \int \int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \]
and
\[ g_\mu(f)(x) = \left( \int \int_{\mathbb{R}^n_+} \left( \frac{t}{t + |x-y|} \right)^{n/2} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \]
which are the Littlewood-Paley operators (see [17]). Let \( H \) be the space
\[ H = \left\{ h : ||h|| = \left( \int_0^\infty |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \right\} \]
or
\[ H = \left\{ h : ||h|| = \left( \int \int_{\mathbb{R}^n_+} |h(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} < \infty \right\}, \]
then, for each fixed \( x \in \mathbb{R}^n \), \( F^A_t(f)(x) \) and \( F^A_t(f)(x, y) \) may be viewed as the mapping from \([0, +\infty)\) to \( H \), and it is clear that
\[ g_\psi^A(f)(x) = ||F^A_t(f)(x)||, \quad g_\mu(f)(x) = ||F_t(f)(x)||, \]
\[ S_\psi^A(f)(x) = ||\chi_{\Gamma(x)} F^A_t(f)(x, y)||, \quad S_\psi(f)(x) = ||\chi_{\Gamma(x)} F_t(f)(y)|| \]
and
\[ g_\mu^A(f)(x) = \left\| \left( \frac{t}{t + |x-y|} \right)^{n/2} F^A_t(f)(x, y) \right\|, \]
\[ g_\mu(f)(x) = \left\| \left( \frac{t}{t + |x-y|} \right)^{n/2} F_t(f)(y) \right\|. \]
It is easily to see that \( g_\psi, S_\psi \) and \( g_\mu \) satisfy the conditions of Theorem 1, 2, 3 and 4, thus the conclusions of Theorem 1, 2, 3 and 4 hold for \( g_\psi^A \) and \( g_\mu^A \), \( S_\psi^A \) and \( S_\psi \), \( g_\psi^A \) and \( g_\mu^A \).

**Application 2.** Marcinkiewicz operator.

Fixed \( 0 \leq \delta < n \), Fix \( \lambda > \max(1, 2n/(n + 2 - 2\delta)) \) and \( 0 < \gamma \leq 1 \). Let \( \Omega \) be homogeneous of degree zero on \( \mathbb{R}^n \) with \( \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0 \). Assume that \( \Omega \in Lip_\gamma(S^{n-1}) \). The Marcinkiewicz multilinear operators are defined by (see [12], [18])
\[ \mu_\Omega^A(f)(x) = \left( \int_0^\infty |F^A_t(f)(x)|^2 \frac{dt}{t^n} \right)^{1/2}, \]
We also define that
\[
\mu^A_S(f)(x) = \left[ \iint_{\Gamma(x)} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}
\]
and
\[
\mu^A_f(x) = \left[ \iint_{R^{n+1}_t} \left( \frac{t}{t + |x - y|} \right)^n |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},
\]
where
\[
F^A_t(f)(x) = \int_{|x - y| \leq t} \prod_{j=1}^l R_m(A_j; x,y) \frac{\Omega(x - y)}{|x - y|^n} \frac{\Omega(y - z)}{|y - z|^n} f(y) dy
\]
and
\[
F_t^A(f)(x,y) = \int_{|y - z| \leq t} \prod_{j=1}^l Q_j(A_j; y,z) \frac{\Omega(y - z)}{|y - z|^n} \frac{\Omega(y - z)}{|y - z|^n} f(z) dz;
\]
The variants of \( \mu^A_{\Omega} \), \( \mu^A_S \) and \( \mu^A_f \) are defined by
\[
\tilde{\mu}^A_{\Omega}(f)(x) = \left( \int_0^\infty |\tilde{F}^A_t(f)(x)|^2 \frac{dt}{t^n} \right)^{1/2},
\]
\[
\tilde{\mu}^A_S(f)(x) = \left[ \iint_{\Gamma(x)} |\tilde{F}^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},
\]
and
\[
\tilde{\mu}^A_f(x) = \left[ \iint_{R^{n+1}_t} \left( \frac{t}{t + |x - y|} \right)^n |\tilde{F}^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2},
\]
where
\[
\tilde{F}^A_t(f)(x) = \int_{|x - y| \leq t} \prod_{j=1}^l Q_m(A_j; x,y) \frac{\Omega(x - y)}{|x - y|^n} \frac{\Omega(y - z)}{|y - z|^n} f(y) dy
\]
and
\[
\tilde{F}^A_t(f)(x,y) = \int_{|y - z| \leq t} \prod_{j=1}^l Q_j(A_j; y,z) \frac{\Omega(y - z)}{|y - z|^n} \frac{\Omega(y - z)}{|y - z|^n} f(z) dz.
\]
Set
\[
F_t(f)(x) = \int_{|x - y| \leq t} \frac{\Omega(x - y)}{|x - y|^n} f(y) dy;
\]
We also define that
\[
\mu_{\Omega}(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^n} \right)^{1/2},
\]
\[
\mu_S(f)(x) = \left( \iint_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}.
\]
and
\[ \mu_\lambda(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \left| F_t(f)(y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]
which are the Marcinkiewicz operators (see [18]). Let \( H \) be the space
\[ H = \left\{ h : \|h\| = \left( \int_{\mathbb{R}^n} |h(t)|^2 \frac{dt}{t^3} \right)^{1/2} < \infty \right\}, \]
or
\[ H = \left\{ h : \|h\| = \left( \int \int_{\mathbb{R}^{n+1}_+} |h(y, t)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} < \infty \right\}. \]
Then, it is clear that
\[ \mu^A_\Omega(f)(x) = ||F^A_t(f)(x)||, \quad \mu^A_S(f)(x) = ||\chi_{\Gamma(x)}F^A_t(f)(x, y)||, \quad \mu^A_\lambda(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} \left| F^A_t(f)(x, y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]
\[ \mu^A_\lambda(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} \left| F_t(f)(y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \]
and
\[ \mu^A_\lambda(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\lambda/2} \left| \chi_{\Gamma(x)}F^A_t(f)(x, y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}. \]
It is easily to see that \( \mu_Q, \mu_S \) and \( \mu_\lambda \) satisfy the conditions of Theorem 1, 2, 3 and 4, thus Theorem 1, 2, 3 and 4 hold for \( \mu^A_\Omega, \mu^A_S, \mu^A_\lambda, \tilde{\mu}^A_\Omega, \tilde{\mu}^A_S, \tilde{\mu}^A_\lambda \).

References


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