A COMMON FIXED POINT THEOREM IN TWO $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract. In this paper, we give some new definitions of $\mathcal{M}$-fuzzy metric spaces and we prove a common fixed point theorem for six mappings under the condition of compatible mappings of first or second type in two complete $\mathcal{M}$-fuzzy metric spaces.

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [20] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [6] and Kramosil and Michalek [9] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [2, 3, 4, 5, 17]. Many authors [8, 12, 15] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [18] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence (see Note 3.13 and Definition 3.15 of [6] also [16, 19]). In this paper, we prove a common fixed point theorem in fuzzy metric spaces for arbitrary $t$-norms and modified definition of Cauchy sequence in George and Veeramani’s sense. There have been a number of generalizations of metric spaces. One such generalization is generalized metric space or $D$-metric space initiated by Dhage [1] in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded $D$-metric spaces. Rhoades [10] generalized Dhage’s contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in $D$-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [14] introduced the concept of $D$-compatibility of maps in $D$-metric space and proved some fixed point theorems using a contractive condition.

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In what follows \((X, D)\) will denote a \(D\)-metric space, \(\mathbb{N}\) the set of all natural numbers, and \(\mathbb{R}^+\) the set of all positive real numbers.

**Definition 1.1.** Let \(X\) be a nonempty set. A generalized metric (or \(D\)-metric) on \(X\) is a function: \(D : X^3 \rightarrow \mathbb{R}^+\) that satisfies the following conditions for each \(x, y, z, a \in X\).

1. \(D(x, y, z) \geq 0,\)
2. \(D(x, y, z) = 0\) if and only if \(x = y = z,\)
3. \(D(x, y, z) = D(p\{x, y, z\})\), (symmetry) where \(p\) is a permutation function,
4. \(D(x, y, z) \leq D(x, y, a) + D(a, z, z).\)

The pair \((X, D)\) is called a generalized metric (or \(D\)-metric) space.

Immediate examples of such a function are
(a) \(D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},\)
(b) \(D(x, y, z) = d(x, y) + d(y, z) + d(z, x).\)

Here, \(d\) is the ordinary metric on \(X\).
(c) If \(X = \mathbb{R}^n\) then we define \(D(x, y, z) = (|x - y|^p + |y - z|^p + |z - x|^p)^{\frac{1}{p}}\) for every \(p \in \mathbb{R}^+.\)
(d) If \(X = \mathbb{R}^+\) then we define \(D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise}. \end{cases}\)

**Remark 1.2.** In a \(D\)-metric space, we prove that \(D(x, x, y) = D(x, y, y)\). For
(i) \(D(x, x, y) \leq D(x, x, x) + D(x, y, y) = D(x, y, y)\) and similarly
(ii) \(D(y, y, x) \leq D(y, y, y) + D(y, x, x) = D(y, x, x)\).
Hence by (i), (ii) we get \(D(x, x, y) = D(x, y, y)\).

Let \((X, D)\) be a \(D\)-metric space. For \(r > 0\) define
\(B_D(x, r) = \{y \in X : D(x, y, y) < r\}\)

**Example 1.3.** Let \(X = \mathbb{R}\). Denote \(D(x, y, z) = |x - y| + |y - z| + |z - x|\) for all \(x, y, z \in \mathbb{R}\). Thus
\(B_D(1, 2) = \{y \in \mathbb{R} : D(1, y, y) < 2\} = \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2).\)

**Definition 1.4.** Let \((X, D)\) be a \(D\)-metric space and \(A \subset X\).

1. If for every \(x \in A\) there exist \(r > 0\) such that \(B_D(x, r) \subset A\), then subset \(A\) is called open subset of \(X\).
2. Subset \(A\) of \(X\) is said to be \(D\)-bounded if there exists \(r > 0\) such that \(D(x, y, y) < r\) for all \(x, y \in A\).
(3) A sequence \( \{x_n\} \) in \( X \) converges to \( x \) if and only if \( D(x_n, x_n, x) = D(x, x, x_n) \to 0 \) as \( n \to \infty \). That is for each \( \epsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that

\[
\forall n \geq n_0 \implies D(x, x, x_n) < \epsilon.
\]

This is equivalent with, for each \( \epsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that

\[
\forall n, m \geq n_0 \implies D(x, x_n, x_m) < \epsilon.
\]

Indeed, if have (\( \ast \)), then

\[
D(x_n, x_m, x) = D(x_n, x, x_m) \leq D(x_n, x, x) + D(x, x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Conversely, set \( m = n \) in (\( \ast \ast \)) we have \( D(x_n, x_n, x) < \epsilon \).

(4) Sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if for each \( \epsilon > 0 \), there exits \( n_0 \in \mathbb{N} \) such that \( D(x_n, x_n, x_m) < \epsilon \) for each \( n, m \geq n_0 \). The \( D \)-metric space \((X, D)\) is said to be complete if every Cauchy sequence is convergent.

Let \( \tau \) be the set of all \( A \subseteq X \) with \( x \in A \) if and only if there exist \( r > 0 \) such that \( B_D(x, r) \subseteq A \). Then \( \tau \) is a topology on \( X \) (induced by the \( D \)-metric \( D \)).

**Lemma 1.5.** Let \((X, D)\) be a \( D \)-metric space. If \( r > 0 \), then ball \( B_D(x, r) \) with center \( x \in X \) and radius \( r \) is open ball.

**Proof.** Let \( z \in B_D(x, r) \), hence \( D(x, z, z) < r \). If set \( D(x, z, z) = \delta \) and \( r' = r - \delta \) then we prove that \( B_D(z, r') \subseteq B_D(x, r) \). Let \( y \in B_D(z, r') \), by triangular inequality we have \( D(x, y, y) = D(y, y, x) \leq D(y, y, z) + D(z, x, x) < r' + \delta = r \).

Hence \( B_D(z, r') \subseteq B_D(x, r) \). That is ball \( B_D(x, r) \) is open ball. \( \square \)

**Lemma 1.6.** Let \((X, D)\) be a \( D \)-metric space. If sequence \( \{x_n\} \) in \( X \) converges to \( x \), then \( x \) is unique.

**Proof.** Let \( x_n \to y \) and \( y \neq x \). Since \( \{x_n\} \) converges to \( x \) and \( y \), for each \( \epsilon > 0 \) there exist \( n_1 \in \mathbb{N} \) such that for every \( n \geq n_1 \implies D(x, x_n, x) < \frac{\epsilon}{2} \) and \( n_2 \in \mathbb{N} \) such that for every \( n \geq n_2 \implies D(y, y, x_n) < \frac{\epsilon}{2} \).

If set \( n_0 = \max\{n_1, n_2\} \), then for every \( n \geq n_0 \) by triangular inequality we have

\[
D(x, x, y) \leq D(x, x, x_n) + D(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Hence \( D(x, x, y) = 0 \) is a contradiction. So, \( x = y \). \( \square \)

**Lemma 1.7.** Let \((X, D)\) be a \( D \)-metric space. If sequence \( \{x_n\} \) in \( X \) converges to \( x \), then sequence \( \{x_n\} \) is a Cauchy sequence.

**Proof.** Since \( x_n \to x \) for each \( \epsilon > 0 \) there exists

\[
n_1 \in \mathbb{N} \text{ such that for every } n \geq n_1 \implies D(x_n, x_n, x) < \frac{\epsilon}{2}
\]

and

\[
n_2 \in \mathbb{N} \text{ such that for every } m \geq n_2 \implies D(x, x_m, x_m) < \frac{\epsilon}{2}
\]
If set \( n_0 = \max\{n_1, n_2\} \), then for every \( n, m \geq n_0 \) by triangular inequality we have
\[
D(x_n, x_n, x_m) \leq D(x_n, x_n, x) + D(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Hence sequence \( \{x_n\} \) is a Cauchy sequence.

**Definition 1.8.** A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions
1. \( * \) is associative and commutative,
2. \( * \) is continuous,
3. \( a * 1 = a \) for all \( a \in [0, 1] \),
4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).

Two typical examples of continuous \( t \)-norm are \( a * b = ab \) and \( a * b = \min(a, b) \).

**Definition 1.9.** A 3-tuple \((X, \mathcal{M}, *)\) is called a \( \mathcal{M} \)-fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( * \) is a continuous \( t \)-norm, and \( \mathcal{M} \) is a fuzzy set on \( X^3 \times (0, \infty) \), satisfying the following conditions for each \( x, y, z, a \in X \) and \( t, s > 0 \),
1. \( \mathcal{M}(x, y, z, t) > 0 \),
2. \( \mathcal{M}(x, y, z, t) = 1 \) if and only if \( x = y = z \),
3. \( \mathcal{M}(x, y, z, t) = \mathcal{M}(p[x, y, z], t) \), (symmetry) where \( p \) is a permutation function,
4. \( \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, s, t) \leq \mathcal{M}(x, y, z, t + s) \),
5. \( \mathcal{M}(x, y, z, t) : (0, \infty) \rightarrow [0, 1] \) is continuous.

**Remark 1.10.** Let \((X, \mathcal{M}, *)\) be a \( \mathcal{M} \)-fuzzy metric space. We prove that for every \( t > 0 \), \( \mathcal{M}(x, y, t, t) = \mathcal{M}(x, y, y, t) \). Because for each \( \epsilon > 0 \) by triangular inequality we have
\[
(i) \quad \mathcal{M}(x, x, y, t+\epsilon) \geq \mathcal{M}(x, x, x, t) \mathcal{M}(x, y, y, t) \mathcal{M}(y, y, y, t)
\]
\[
(ii) \quad \mathcal{M}(y, y, x, t+\epsilon) \geq \mathcal{M}(y, y, y, t) \mathcal{M}(y, y, x, t) \mathcal{M}(y, x, y, t).
\]

By taking limits of (i) and (ii) when \( \epsilon \rightarrow 0 \), we obtain \( \mathcal{M}(x, y, t, t) = \mathcal{M}(x, y, y, t) \).

Let \((X, \mathcal{M}, *)\) be a \( \mathcal{M} \)-fuzzy metric space. For \( t > 0 \), the open ball \( B_{\mathcal{M}}(x, r, t) \) with center \( x \in X \) and radius \( 0 < r < 1 \) is defined by
\[
B_{\mathcal{M}}(x, r, t) = \{ y \in X : \mathcal{M}(x, y, y, t) > 1 - r \}.
\]

A subset \( A \) of \( X \) is called open set if for each \( x \in A \) there exist \( t > 0 \) and \( 0 < r < 1 \) such that \( B_{\mathcal{M}}(x, r, t) \subseteq A \). A sequence \( \{x_n\} \) in \( X \) converges to \( x \) if and only if \( \mathcal{M}(x, x, x_n, t) \rightarrow 1 \) as \( n \rightarrow \infty \), for each \( t > 0 \). It is called a Cauchy sequence if for each \( 0 < \epsilon < 1 \) and \( t > 0 \), there exist \( n_0 \in \mathbb{N} \) such that \( \mathcal{M}(x_n, x_m, x_n, t) > 1 - \epsilon \) for each \( n, m \geq n_0 \). The \( \mathcal{M} \)-fuzzy metric \((X, \mathcal{M}, *)\) is said to be complete if every Cauchy sequence is convergent.

**Example 1.11.** Let \( X \) be a nonempty set and \( D \) be the \( D \)-metric on \( X \). Denote \( a * b = ab \) for all \( a, b \in [0, 1] \). For each \( t \in [0, \infty] \), define
\[
\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}
\]
for all \( x, y, z \in X \). It is easy to see that \((X, \mathcal{M}, \ast)\) is a \(\mathcal{M}\)-fuzzy metric space.

**Lemma 1.12.** Let \((X, \mathcal{M}, \ast)\) be a fuzzy metric space. If we define \(\mathcal{M} : X^3 \times (0, \infty) \rightarrow [0, 1]\) by
\[
\mathcal{M}(x, y, z, t) = \mathcal{M}(x, y, t) \ast \mathcal{M}(y, z, t) \ast \mathcal{M}(z, x, t)
\]
for every \( x, y, z \) in \( X \), then \((X, \mathcal{M}, \ast)\) is a \(\mathcal{M}\)-fuzzy metric space.

**Proof.**
(1) It is easy to see that for every \( x, y, z \in X \), \( \mathcal{M}(x, y, z, t) > 0 \forall t > 0 \).
(2) \( \mathcal{M}(x, y, z, t) = 1 \) if and only if \( \mathcal{M}(x, y, t) = \mathcal{M}(y, z, t) = \mathcal{M}(z, x, t) = 1 \) if and only if \( x = y = z \).
(3) \( \mathcal{M}(x, y, z, t) = \mathcal{M}(p(x, y, z), t) \), where \( p \) is a permutation function.
(4) \[
\begin{align*}
\mathcal{M}(x, y, z, t + s) & = \mathcal{M}(x, y, t + s) \ast \mathcal{M}(y, z, t + s) \ast \mathcal{M}(z, x, t + s) \\
& \geq \mathcal{M}(x, y, t) \ast \mathcal{M}(y, a, t) \ast \mathcal{M}(a, z, s) \ast \mathcal{M}(z, a, s) \ast \mathcal{M}(a, x, t) \\
& = \mathcal{M}(x, y, a, t) \ast \mathcal{M}(a, z, s) \ast \mathcal{M}(z, a, s) \ast \mathcal{M}(z, z, s) \\
& \geq \mathcal{M}(x, y, a, t) \ast \mathcal{M}(a, z, z, s) \quad \text{for every } t, s > 0.
\end{align*}
\]

\( \square \)

**Definition 1.13.** Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space, then \(\mathcal{M}\) is called of **first type** if for every \( x, y \in X \) we have
\[
\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, z, t)
\]
for every \( z \in X \).

Also it is called of **second type** if for every \( x, y, z \in X \) we have
\[
\mathcal{M}(x, y, z, t) = \mathcal{M}(x, y, t) \ast \mathcal{M}(y, z, t) \ast \mathcal{M}(z, x, t).
\]

Let \( a \ast b = \min(a, b) \) for every \( a, b \in [0, 1] \) in this case it is easy to see that, if \(\mathcal{M}\) is second type then \(\mathcal{M}\) is first type.

**Example 1.14.** If we define \(\mathcal{M}(x, y, z, t) = \frac{t}{t + D(x, y, z)}\) where \( D(x, y, z) = d(x, y) + d(y, z) + d(z, x) \), or define
\[
\mathcal{M}(x, y, z, t) = \begin{cases} 
\frac{1}{t + \max\{x, y, z\}} & \text{if } x = y = z, \\
\frac{t}{t + \max\{x, y, z\}} & \text{otherwise},
\end{cases}
\]
then \(\mathcal{M}\) is first type.

If \((X, \mathcal{M}, \ast)\) is a fuzzy metric and \( M(x, y, t) = \frac{t}{t + d(x, y)} \), then
\[
\mathcal{M}(x, y, z, t) = \frac{t}{t + d(x, y)} \ast \frac{t}{t + d(y, z)} \ast \frac{t}{t + d(z, x)}
\]
is second type.
Remark 1.15. Let \((X, \mathcal{M}, *)\) be a \(\mathcal{M}\)-fuzzy metric space. If \(\mathcal{M}\) is second type, sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if \(\mathcal{M}(x, x, x, t) \to 1\) or if and only if \(M(x, x, t) \to 1\). For

\[
\mathcal{M}(x, x, x, t) = \mathcal{M}(x, x, t) * \mathcal{M}(x, x, t) * \mathcal{M}(x, x, t)
\]

= \(M(x, x, t) * M(x, x, t)\).

2. The main results

Lemma 2.1. Let \((X, \mathcal{M}, *)\) be a \(\mathcal{M}\)-fuzzy metric space. Then \(\mathcal{M}(x, y, z, t)\) is nondecreasing with respect to \(t\), for all \(x, y, z\) in \(X\).

Proof. By Definition 1.9(4) for each \(x, y, z, a \in X\) and \(t, s > 0\) we have

\[
\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, s) \leq \mathcal{M}(x, y, z, s + t).
\]

If set \(a = z\) we get \(\mathcal{M}(x, y, z, t) * M(z, z, s) \leq \mathcal{M}(x, y, z, t + s)\), that is, \(\mathcal{M}(x, y, z, t + s) \geq \mathcal{M}(x, y, z, t)\).

Definition 2.2. Let \((X, \mathcal{M}, *)\) be a \(\mathcal{M}\)-fuzzy metric space. \(\mathcal{M}\) is said to be continuous function on \(X^3 \times (0, \infty)\) if

\[
\lim_{n \to \infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t).
\]

Whenever a sequence \(\{(x_n, y_n, z_n, t_n)\}\) in \(X^3 \times (0, \infty)\) converges to a point \((x, y, z, t) \in X^3 \times (0, \infty)\) i.e.,

\[
\lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y, \lim_{n \to \infty} z_n = z \quad \text{and} \quad \lim_{n \to \infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).
\]

Lemma 2.3. Let \((X, \mathcal{M}, *)\) be a \(\mathcal{M}\)-fuzzy metric space. Then \(M\) is continuous function on \(X^3 \times (0, \infty)\).

Proof. Let \(x, y, z \in X\) and \(t > 0\), and let \((x_n', y_n', z_n', t_n')_n\) be a sequence in \(X^3 \times (0, \infty)\) that converges to \((x, y, z, t)\). Since \((M(x_n', y_n', z_n', t_n'))_n\) is a sequence in \((0, 1]\), there is a subsequence \((x_n, y_n, z_n, t_n)_n\) of sequence \((x_n', y_n', z_n', t_n')_n\) such that sequence \((M(x_n, y_n, z_n, t_n))_n\) converges to some point of \([0, 1]\). Fix \(\delta > 0\) such that \(\delta < \frac{t}{4}\). Then, there is \(n_0 \in \mathbb{N}\) such that \(|t - t_{n_0}| < \delta\) for every \(n \geq n_0\). Hence,

\[
\mathcal{M}(x_n, y_n, z_n, t_n) \geq \mathcal{M}(x_n, y_n, z_n, t - \delta) \geq \mathcal{M}(x_n, y_n, z, t - \frac{4\delta}{3}) \ast \mathcal{M}(z, z_n, z_n, z_n, \frac{\delta}{3})
\]

\[
\geq \mathcal{M}(x_n, z, y, t - \frac{5\delta}{3}) \ast \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) \ast \mathcal{M}(z, z_n, z_n, \frac{\delta}{3})
\]

\[
\geq \mathcal{M}(z, y, x, t - 2\delta) \ast \mathcal{M}(x, x_n, x_n, \frac{\delta}{3}) \ast \mathcal{M}(y, y_n, y_n, \frac{\delta}{3}) \ast \mathcal{M}(z, z_n, z_n, \frac{\delta}{3})
\]
and
\[ M(x, y, z, t + 2\delta) \geq M(x, y, z, t_n + \delta) \geq M(x, y, z, t_n + \frac{2\delta}{3}) \ast M(z_n, z, \frac{\delta}{3}) \]
\[ \geq M(z_n, y_n, t_n + \frac{\delta}{3}) \ast M(y_n, y, \frac{\delta}{3}) \ast M(z_n, z, \frac{\delta}{3}) \]
\[ \geq M(z_n, y_n, x_n, t_n) \ast M(x_n, x, \frac{\delta}{3}) \ast M(y_n, y, \frac{\delta}{3}) \ast M(z_n, z, \frac{\delta}{3}) \]
for all \( n \geq n_0 \). By taking limits when \( n \to \infty \), we obtain
\[ \lim_{n \to \infty} M(x_n, y_n, z_n, t_n) \geq M(x, y, z, t - 2\delta) \ast 1 \ast 1 = M(x, y, z, t - 2\delta) \]
and
\[ M(x, y, z, t + 2\delta) \geq M(x_n, y_n, z_n, t_n)1 \ast 1 = \lim_{n \to \infty} M(x_n, y_n, z_n, t_n), \]
respectively. So, by continuity of the function \( t \to M(x, y, z, t) \), we immediately deduce that
\[ \lim_{n \to \infty} M(x_n, y_n, z_n, t_n) = M(x, y, z, t). \]
Therefore \( M \) is continuous on \( X^3 \times (0, \infty) \). \( \Box \)

Henceforth, we assume that \( \ast \) is a continuous t-norm on \([0,1]\) such that for every \( \mu \in (0,1) \), there is a \( \lambda \in (0,1) \) such that
\[ (1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda) \geq 1 - \mu \]

Lemma 2.4. Let \((X, M, \ast)\) be a \( M \)-fuzzy metric space. If we define \( E_{\lambda, M} : X^3 \to \mathbb{R}^+ \cup \{0\} \) by
\[ E_{\lambda, M}(x, y, z) = \inf\{t > 0 : M(x, y, z, t) > 1 - \lambda\} \]
for every \( \lambda \in (0,1) \), then
(i) for each \( \mu \in (0,1) \) there exists \( \lambda \in (0,1) \) such that
\[ E_{\mu, M}(x_1, x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n) \]
for any \( x_1, x_2, \ldots, x_n \in X \),
(ii) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is convergent in \( M \)-fuzzy metric space \((X, M, \ast)\) if and only if \( E_{\lambda, M}(x_n, x_n, x) \to 0 \). Also the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is Cauchy sequence if and only if it is Cauchy with \( E_{\lambda, M} \).

Proof. (i). For every \( \mu \in (0,1) \), we can find a \( \lambda \in (0,1) \) such that
\[ (1 - \lambda) \ast (1 - \lambda) \ast \cdots \ast (1 - \lambda) \geq 1 - \mu \]
by triangular inequality we have
\[ M(x_1, x_1, x_n, E_{\lambda, M}(x_1, x_1, x_2) + E_{\lambda, M}(x_2, x_2, x_3) + \cdots
+ E_{\lambda, M}(x_{n-1}, x_{n-1}, x_n) + n\delta) \]
\[ \geq M(x_1, x_1, x_2, E_{\lambda, M}(x_1, x_1, x_2) + \delta) \cdots
* M(x_{n-1}, x_{n-1}, x_n, E_{\lambda, M}(x_{n-1}, x_{n-1}, x_n) + \delta) \]
\[ \geq (1 - \lambda) \cdots (1 - \lambda) \geq 1 - \mu \]
for very \( \delta > 0 \), which implies that

\[ E_{\mu, M}(x_1, x_1, x_n) \leq E_{\lambda, M}(x_1, x_1, x_2) + E_{\lambda, M}(x_2, x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_{n-1}, x_n) + n\delta. \]

Since \( \delta > 0 \) is arbitrary, we have

\[ E_{\mu, M}(x_1, x_1, x_n) \leq E_{\lambda, M}(x_1, x_1, x_2) + E_{\lambda, M}(x_2, x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_{n-1}, x_n). \]

(ii). Note that since \( M \) is continuous in its third place and \( E_{\lambda, M}(x, x, y) = \inf \{ t > 0 : M(x, x, y, t) > 1 - \lambda \} \).

Hence, we have

\[ M(x_n, x, x, \eta) > 1 - \lambda \iff E_{\lambda, M}(x_n, x, x) < \eta \]
for every \( \eta > 0 \). \( \square \)

**Lemma 2.5.** Let \((X, M, \ast)\) be a \(M\)-fuzzy metric space. If
\[ M(x_n, x_n, x_{n+1}, t) \geq M(x_0, x_0, x_1, k^n t) \]
for some \( k > 1 \) and for every \( n \in \mathbb{N} \). Then sequence \( \{x_n\} \) is a Cauchy sequence.

**Proof.** For every \( \lambda \in (0, 1) \) and \( x_n, x_{n+1} \in X \), we have

\[ E_{\lambda, M}(x_n, x_n, x_{n+1}) = \inf \{ t > 0 : M(x_n, x_n, x_{n+1}, t) > 1 - \lambda \} \]
\[ \leq \inf \{ t > 0 : M(x_0, x_0, x_1, k^n t) > 1 - \lambda \} \]
\[ = \inf \{ \frac{t}{k^n} > 0 : M(x_0, x_0, x_1, t) > 1 - \lambda \} \]
\[ = \frac{1}{k^n} \inf \{ t > 0 : M(x_0, x_0, x_1, t) > 1 - \lambda \} \]
\[ = \frac{1}{k^n} E_{\lambda, M}(x_0, x_0, x_1). \]
By Lemma 2.4, for every $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that
\[
E_{\mu,M}(x_n, x_n, x_m) \leq E_{\lambda,M}(x_n, x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}, x_{m-1}, x_{m}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m)
\]
\[
\leq \frac{1}{k^n} E_{\lambda,M}(x_0, x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,M}(x_0, x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,M}(x_0, x_0, x_1)
\]
\[
= E_{\lambda,M}(x_0, x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0.
\]
Hence sequence $\{x_n\}$ is Cauchy sequence. □

A class of implicit relation

Let $\Phi$ denote a family of mappings such that each $\phi \in \Phi$, $\phi : [0, 1] \longrightarrow [0, 1]$, such that $\phi$ is continuous and $\phi(s) > s$ for every $s \in [0, 1]$.

**Theorem 2.6.** Let $(X, M, \ast)$ and $(Y, N, \circ)$ be two complete $M$ and $N$-fuzzy metric spaces, respectively where $M$ and $N$ are first or second type. If $A, B, C, T, S, R$ be three mappings of $Y$ to $X$ and $T, S, R$ be three mappings of $X$ to $X$ such that satisfies the following conditions:

(i) $M(SAx, TBo, RCx'', t) \geq \phi(M(x, x', x'', k_1 t))$, for every $x, x', x'' \in X$ some $k_1 > 1$ and $\phi \in \Phi$, 

(ii) $N(CTy, ARy', BSy'', t) \geq \psi(N(y, y', y'', k_2 t))$, for every $y, y', y'' \in Y$ some $k_2 > 1$ and $\psi \in \Phi$.

If at least $A, B, C, T, S$ or $R$ be continuous mapping, then there exist a unique point $z \in X$ and $w \in Y$, such that $SAz = TBz = RCz = z$ and $ARw = BSz = CTw = w$. Moreover,

\[
Sw = Tw = Rw = z \quad \quad Az = Bz = Cz = w.
\]

**Proof.** Let $x_0 \in X$ be an arbitrary point in $X$, define

$Ax_0 = y_1$, \quad $Sy_1 = x_1$, \quad $Bx_1 = y_2$, \quad $Ty_2 = x_2$, \quad $Cx_2 = y_3$, \quad and \quad $Ry_3 = x_3$.

So by induction, for $n = 0, 1, 2, \ldots$ we have

$Ax_{3n} = y_{3n+1}, Sy_{3n+1} = x_{3n+1}, Bx_{3n+1} = y_{3n+2}, $ \quad $Ty_{3n+2} = x_{3n+2}, Cx_{3n+2} = y_{3n+3}, Ry_{3n+3} = x_{3n+3}$.

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are a Cauchy sequence in $X$ and $Y$ respectively. Let

\[
d_n(t) = M(x_n, x_{n+1}, x_{n+2}, t).
\]
Now, for $3n$, we get

\[ d_{3n}(t) = M(x_{3n}, x_{3n+1}, x_{3n+2}, t) \]
\[ = M(Ry_{3n}, Sy_{3n+1}, Ty_{3n+2}, t) \]
\[ = M(RCx_{3n-1}, SAx_{3n}, TBx_{3n+1}, t) \]
\[ = M(SAx_{3n}, TBx_{3n+1}, RC_{3n-1}, t) \]
\[ = \phi(M(x_{3n}, 3n+1, 3n+2, k_t)) \]
\[ \geq M(x_{3n-1}, x_{3n}, x_{3n+1}, k_t) \]
\[ = d_{3n-1}(k_t). \]

For $3n + 1$, we have

\[ d_{3n+1}(t) = M(x_{3n+1}, x_{3n+2}, x_{3n+3}, t) = M(Sy_{3n+1}, Ty_{3n+2}, Ry_{3n+3}, t) \]
\[ = M(SAx_{3n}, TBx_{3n+1}, RC_{3n}, t) \]
\[ = \phi(M(x_{3n}, x_{3n+1}, x_{3n+2}, k_t)) \]
\[ = M(x_{3n}, x_{3n+1}, x_{3n+2}, k_t) = d_{3n}(k_t). \]

Also, for $3n + 2$, we get

\[ d_{3n+2}(t) = M(x_{3n+2}, x_{3n+3}, x_{3n+4}, t) \]
\[ = M(y_{3n}+2, Ry_{3n+3}, Sy_{3n+4}, t) \]
\[ = M(TBx_{3n+1}, RCx_{3n+2}, SAx_{3n+3}, t) \]
\[ = \phi(M(x_{3n+1}, x_{3n+2}, x_{3n+3}, k_t)) \]
\[ = M(x_{3n+1}, x_{3n+2}, x_{3n+3}, k_t) = d_{3n+1}(k_t). \]

Hence for every $n \in \mathbb{N}$ we have $d_n(t) \geq d_{n-1}(k_t)$. That is,

\[ d_n(t) = M(x_n, x_{n+1}, x_{n+1}, t) \]
\[ \geq M(x_{n-1}, x_n, x_{n+1}, k_t) \]
\[ \geq \cdots \geq M(x_0, x_1, x_2, k_1^t). \]

Since $M$ is a first or second type, hence by Remark 1.15 \{x_n\} is Cauchy and the completeness of $X$, \{x_n\} converges to $z$ in $X$. That is, $\lim_{n \to \infty} x_n = z$.

Let

\[ L_n(t) = N(y_n, y_{n+1}, y_{n+2}, t). \]

Now, for $3n$, we get

\[ L_{3n}(t) = N(y_{3n}, y_{3n+1}, y_{3n+2}, t) = N(Cx_{3n-1}, Ax_{3n}, Bx_{3n+1}, t) \]
\[ = N(CTy_{3n-1}, ARy_{3n}, BSy_{3n+1}, t) = N(SAx_{3n}, TBx_{3n+1}, RC_{3n-1}, t) \]
\[ \geq \psi(N(y_{3n-1}, y_{3n}, y_{3n+1}, k_2t)) \]
\[ \geq N(y_{3n-1}, y_{3n}, y_{3n+1}, k_2t) = L_{3n-1}(k_2t). \]
For $3n + 1$, we have
\[
L_{3n+1}(t) = N(y_{3n+1}, y_{3n+2}, y_{3n+3}, t) = N(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}, t)
\]
\[
= N(ARy_{3n}, BSy_{3n+1}, CTy_{3n+2}, t)
\]
\[
\geq \psi(N(y_{3n+1}, y_{3n+2}, k_{2}t))
\]
\[
= N(y_{3n+1}, y_{3n+1}, y_{3n+2}, k_{2}t) = L_{3n}(k_{2}t).
\]

Also, for $3n + 2$, we get
\[
L_{3n+2}(t) = N(y_{3n+2}, y_{3n+3}, y_{3n+4}, t)
\]
\[
= N(Bx_{3n+2}, Cx_{3n+3}, Ax_{3n+4}, t)
\]
\[
= N(BSy_{3n+1}, CTy_{3n+2}, ARy_{3n+3}, t)
\]
\[
\geq \psi(N(y_{3n+1}, y_{3n+2}, y_{3n+3}, k_{2}t))
\]
\[
= N(y_{3n+1}, y_{3n+2}, y_{3n+3}, k_{2}t) = L_{3n+1}(k_{2}t).
\]

Hence for every $n \in \mathbb{N}$ we have $L_{n}(t) \geq L_{n-1}(k_{2}t)$. That is,
\[
L_{n}(t) = N(y_{n}, y_{n+1}, y_{n+1}, t)
\]
\[
\geq N(y_{n-1}, y_{n}, y_{n+1}, k_{2}t) \geq \cdots \geq M(y_{0}, y_{1}, y_{2}, k_{2}t).
\]

Since $N$ is a first or second type, hence by Remark 1.15 \{y_{n}\} is Cauchy and the completeness of $Y$, \{y_{n}\} converges to $w$ in $Y$. That is, $\lim_{n \to \infty} y_{n} = w$.

Let $A$ is continuous, hence $\lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} Ax_{3n} = A \lim_{n \to \infty} x_{3n} = Az = w$. Now, we prove that $SAz = z$. For by (i), we have
\[
M(SAz, TBx_{3n+1}, RCx_{3n+2}, t) \geq \phi(M(z, x_{3n+1}, x_{3n+2}, k_{1}t))
\]

On making $n \to \infty$ we get
\[
M(SAz, z, z, t) \geq \phi(M(z, z, z, k_{1}t)) = \phi(1) = 1.
\]

Thus $Sw = SAz = z$. Now, we prove that $Bz = w$ for
\[
N(CTy_{3n-1}, ARy_{3n}, BSy_{3n}, w, t) \geq \psi(N(y_{3n-1}, y_{3n}, w, k_{2}t)).
\]

Thus
\[
N(y_{3n}, y_{3n+1}, BSy_{3n}, w, t) \geq \psi(N(y_{3n-1}, y_{3n}, w, k_{2}t)).
\]

As $n \to \infty$ we have
\[
N(w, w, BSy_{3n}, w, t) \geq \psi(N(w, w, w, k_{2}t)) = \psi(1) = 1.
\]

Therefore, $BSy_{3n} = Bz = w$. Again, replacing $y$ by $y_{3n-1}$, $y'$ by $w$ and $y''$ by $w$ in (i), we have
\[
N(CTy_{3n-1}, ARw, BSy_{3n}, t) = N(y_{3n}, ARw, BSy_{3n}, t) \geq \psi(N(y_{3n-1}, w, w, k_{2}t)).
\]

On making $n \to \infty$ we get
\[
N(w, ARw, w, t) \geq \psi(N(w, w, w, k_{2}t)) = \psi(1) = 1.
\]

Thus $ARw = w$. So
\[
N(CTw, ARw, BSy_{3n}, w, t) \geq \psi(N(w, w, w, k_{2}t)) = 1.
\]
Therefore, $CTw = ARw = BSw = w$. Again, replacing $x$ by $z$, $x'$ by $z$ and $x''$ by $x_{3n+1}$ in (i), we have

$$M(RCz, SAz, TBx_{3n+1}, t) \geq \phi(M(z, z, x_{3n+1}, t)).$$

On making $n \to \infty$ we get

$$M(RCz, z, z, t) \geq \phi(M(z, z, z, k_1 t)) = 1.$$

Therefore, $RCz = z$. Now, we prove that $TBz = z$ for

$$M(RCz, SAz, TBz, t) \geq \phi(M(z, z, z, k_1 t)) = 1.$$

That is, $TBz = Tw = z$. Hence $TBz = RCz = SAz = z$.

Now, we have $Cz = CTw = w$. So $Rw = RCz = z$. Hence $TAz = RCz = SAz = z$ and $CTw = ARw = BSw = w$.

Therefore $Az = Bz = Cz = w$ and $Sw = Tw = Rw = z$.

Uniqueness, let $z'$ be another common fixed point of $A, B, C$. If $M(z, z, z', t) < 1$, then

$$M(z, z, z', t) = M(TAz, RCz, SAz', t)) \geq \phi(M(z, z, z', k_1 t))$$

is a contradiction. Therefore, $z = z'$ is the unique common fixed point of self-maps $A, B, C$. Similarly we prove that $w$ is unique. Let $w'$ be another common fixed point of $R, S, T$. If $N(w, w, w', t) < 1$, then

$$N(w, w, w', t) = N(CTw, ARw, BSw', t)) \geq \psi(N(w, w, w', k_2 t))$$

is a contradiction. Therefore, $w = w'$ is the unique common fixed point of self-maps $T, R, S$. \hfill \Box

**Example 2.7.** Let $X = [0, 1]$, $Y = [1, 2]$. If $S, T, R : [1, 2] \to [0, 1]$ defined

$$Ty = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ 0 & \text{if } y \text{ is irrational.} \end{cases} \quad Ry = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ \frac{1}{2} & \text{if } y \text{ is irrational.} \end{cases}$$

$$Sy = \begin{cases} 1 & \text{if } y \text{ is rational,} \\ \frac{1}{3} & \text{if } y \text{ is irrational.} \end{cases}$$

Moreover, if $A, B, C : [0, 1] \to [1, 2]$, defined $Ax = 2$ and

$$Bx = \begin{cases} 2 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases} \quad Cx = \begin{cases} 2 & \text{if } x \text{ is rational,} \\ \frac{3}{4} & \text{if } x \text{ is irrational.} \end{cases}$$
Let $M, N, \phi$ and $\psi$ be choice, such that $A, B, C$ and $T, R, S$ satisfying in the above theorem. Then it is easy to see that $A1 = B1 = C1 = 2$ and $T2 = S2 = R2 = 1$. Hence

$$BS2 = AR2 = CT2 = 2 \quad \text{and} \quad TB1 = SA1 = RC1 = 1.$$ 

**Corollary 2.8.** Let $(X, M, *)$ and $(Y, N, \diamond)$ be two complete $M$ and $N$-fuzzy metric spaces, respectively where $M$ and $N$ are first or second type. If $f$ be a mapping of $X$ to $Y$ and $g$ be a mapping of $Y$ to $X$ such that satisfies the following conditions:

(i) $M(gfx, gfx', gfx'', t) \geq \phi(M(x, x', x'', k1t))$, for every $x, x', x'' \in X$ some $k1 > 1$ and $\phi \in \Phi$,

(ii) $N(fgy, fgy', fgy'', t) \geq \psi(N(y, y', y'', k2t))$, for every $y, y', y'' \in Y$ some $k2 > 1$ and $\psi \in \Phi$.

If at least $f$ or $g$ be continuous mapping, then there exist a unique point $z \in X$ and $w \in Y$, such that $gfz = z$ and $fgw = w$. Moreover,

$$gw = z \quad \text{and} \quad fz = w.$$ 

Proof. It is enough set $A = B = C = f$ and $R = S = T = g$ in Theorem 2.6. □

**References**


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