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Abstract. We define the non-associative algebra $W(n, m, m+s)$ and we show that it is simple. We find the non-associative algebra automorphism group $\text{Aut}_{\text{non}}(W(1,0,0))$ of $W(1,0,0)$. Also we find that any derivation of $W(1,0,0)$ is a scalar derivation in this paper.

1. Preliminaries

Let $N$ be the set of all non-negative integers and $Z$ be the set of all integers. Let $F$ be a field of characteristic zero. Let $F^*$ be the multiplicative group of non-zero elements of $F$. The non-associative algebra $W(n, m, m+s)$ is the vector space spanned by

$$\{e^{a_1x_1} \cdots e^{a_nx_n} x_1^{i_1} \cdots x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u | a_1, \ldots, a_n, i_1, \ldots, i_m \in Z, i_{m+1}, \ldots, i_{m+s} \in N\}$$

with the obvious addition and the multiplication $*$ where $\partial_u$ is the usual partial derivative with respect to $x_u$, $1 \leq u \leq \max\{n, m+s\}$ [2], [8], [9]. The non-associative algebra $W(n, m, m+s)$ is a subalgebra of the algebra in the papers [2], [3], [8]. For an element $l$ in an algebra $A$, $l$ is full, if an ideal containing $l$ is $A$. The matrix ring $M_{m+s}(F)$ is imbedded in the non-associative algebra $W(n, m, m+s)$. The matrix ring $M_n(F)$ is not imbedded in $W(n,0,0)$. The non-associative algebra $W(n,0,0)$ has neither a right nor a left multiplicative identity element. Note that the definition of a non-associative algebra in this paper is a little different from the definition of the non-associative algebras in the papers [2], [8], [9], because of some results. Similarly to the non-associative algebra $W(n, m, m+s)$, we can define the non-associative algebra $W(n^+,0,s)$ spanned by $\{e^{a_1x_1} \cdots e^{a_nx_n} x_{n+1}^{i_{n+1}} \cdots x_{n+s}^{i_{n+s}} \partial_u | a_1, \ldots, a_n \in Z, i_{n+1}, \ldots, i_{n+s} \in N\}$.
The non-associative algebra \( W(n, m, m + s) \) is Lie-admissible, since \( W(n, m, m + s) \) is a Lie algebra with respect to the commutator \([,]\) of \( W(n, m, m + s) \). The non-associative algebra \( W(n, m, m + s) \) has idempotents.

2. Simplicity of \( W(n, m, m + s) \)

Even if the non-associative algebra \( W(n, m, m + s) \) has right annihilators, we have the following results.

Remark 1. An (non-associative, Lie, or associative) algebra \( A \) is simple if and only if every element of the (non-associative, Lie, or associative) algebra \( A \) is full.

Lemma 1. For any \( \partial_u, 1 \leq u \leq m + s \), in the non-associative algebra \( W(n, m, m + s) \), \( \partial_u \) is full.

Proof. Let \( I \) be a non-zero ideal of the non-associative algebra \( W(n, m, m + s) \) which contains \( \partial_u \) in the lemma. For any basis element \( e_{a_1}x_1 \cdots e_{a_n}x_n \partial_v \) of \( W(n, m, m + s) \) with \( a_u \neq 0 \),
\[
\partial_u e_{a_1}x_1 \cdots e_{a_n}x_n \partial_v = a_u e_{a_1}x_1 \cdots e_{a_n}x_n \partial_v \in I
\]
This implies that by appropriate inductions on \( x_1, \ldots, x_{m+s} \), we can prove that \( e_{a_1}x_1 \cdots e_{a_n}x_n x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_v \in I \). This implies that \( W(n, m, m + s) \subset I \), i.e., \( W(n, m, m + s) = I \). This implies that \( \partial_u \) is full. Therefore we have proven the lemma.

Theorem 1. The non-associative algebra \( W(n, m, m + s) \) is simple.

Proof. Let \( I \) be a non-zero ideal of the non-associative algebra \( W(n, m, m + s) \). Without loss of generality, we can assume that \( n \leq m + s \). By Lemma 1, we know that \( \partial_u, 1 \leq u \leq m + n \), is full. It is standard to prove that \( \partial_u \in I \). By Remark 1 and Lemma 1, this completes the proof of the theorem.

Corollary 1. The non-associative algebra \( W(n, m, m + s) \) is simple.

Proof. The proof of the corollary is straightforward by Theorem 1. Thus the proof is omitted.

Theorem 2. The Lie algebra \( W(n, m, m + s) \) is simple.

Proof. Since every element of the Lie algebra \( W(n, m, m + s) \) is full, the proof of the theorem is straightforward by Theorem 1. So let omit it.

Corollary 2. The Lie algebras \( W(n^+, 0, s) \) and \( W(0, m, m + s) \) are simple.
Proof. The proof of the corollary is straightforward by Theorem 2, so omitted. □

The Lie algebra $W(n, m, m+s, m+s, m+s+1)$ is called the Witt type Lie algebra [12]. The Lie algebra $W(1, 0, 0)$ is the well known centerless Virasoro algebra [7].

It is easy to prove that the non-associative algebra $W(n, m, m+s, m+s+1)$ is simple.

3. Automorphism group $\text{Aut}_{\text{non}}(W(1, 0, 0))$

Note that by Corollary 1, the non-associative algebra $W(1, 0, 0)$ spanned by $\{e^{ax}\partial | a \in \mathbb{Z}\}$ is simple.

Example 1. The Lie algebra $sl_2(F)$ is isomorphic to the Lie subalgebra of $W(0, 1, 0)$ (resp.$W(1, 0, 0)$) spanned by $\{x^{k+2}\partial, x\partial, x^{-k}\partial\}$ (resp. $\{e^{-ax}\partial, \partial, e^{ax}\partial\}$) where $k, a \in \mathbb{N}$.

Proposition 1. For any non-associative algebra endomorphism $\theta$ of $W(1, 0, 0)$, if $\theta$ is non-zero, then $\theta$ is injective.

Proof. Let $\theta$ be a non-associative algebra endomorphism $\theta$ of $W(1, 0, 0)$. $\text{Ker}(\theta)$ is an ideal of $W(1, 0, 0)$. By Corollary 1, either $\text{Ker}(\theta) = 0$ holds or $\text{Ker}(\theta) = W(1, 0, 0)$ holds. Since $\theta$ is not the zero map, $\text{Ker}(\theta) = 0$. This implies that $\theta$ is injective. So we have proven the proposition. □

Note 1. For any basis element $e^{ax}\partial$ of $W(1, 0, 0)$, if we define $F$-linear maps $\theta_{+d_1}$ and $\theta_{-d_2}$ of $W(1, 0, 0)$, as follows:

$\theta_{+d_1}(e^{(k)x}\partial) = d_1^k e^{(k)x}\partial$

and

$\theta_{-d_2}(e^{kx}\partial) = d_2^k e^{-kx}\partial$

then $\theta_{+d_1}$ and $\theta_{-d_2}$ can be linearly extended to non-associative algebra automorphisms of $W(1, 0, 0)$ where $d_1, d_2 \in F^*$.

Lemma 2. For any non-associative algebra automorphism $\theta$ of $W(1, 0, 0)$, $\theta(\partial) = c\partial$ holds where $c$ is a non-zero scalar.

Proof. Let $\theta$ be the non-associative algebra automorphism $\theta$ of $W(1, 0, 0)$ in the lemma. Since $\partial$ is a basis element of the right annihilator of $W(1, 0, 0)$, $\partial$ is invariant under any automorphism of $W(1, 0, 0)$. This implies that $\theta(\partial) = c\partial$ holds where $c$ is a non-zero scalar. □
Lemma 3. For any $\theta$ in the non-associative algebra automorphism group \( \text{Aut}_{\text{non}}(W(1,0,0)) \) of \( W(1,0,0) \), $\theta$ is either $\theta_{+,-d_1}$ or $\theta_{-,-d_2}$ in Note 1 where $d_1, d_2 \in F^*$. 

Proof. Let $\theta$ be the non-associative algebra automorphism of $\overline{W}(1,0,0)$ in the lemma. By Lemma 2, $\theta(\partial) = c\partial$ holds where $c$ is a non-zero scalar. By Lemma 2 and since $\partial$ is a left identity of $e^x\partial$, we have that

\[
(1) \quad c\partial \ast \theta(e^x\partial) = \theta(e^x\partial).
\]

This implies that $\theta(e^x\partial)$ can be written as follows:

\[
(2) \quad \theta(e^x\partial) = C(b_1)e^{b_1x}\partial + \cdots + C(b_t)e^{b_tx}\partial,
\]

where $C(b_1), \ldots, C(b_t) \in F$ and $b_1 > \cdots > b_t$. By (1) and (2), we have that $c = b_1 = 1$. This implies that either $c = 1$ holds or $c = 1 = -1$ holds.

**Case I.** Let us assume that $c = b_1 = 1$ holds. Let us put $\theta(\partial) = \partial$ and $\theta(e^x\partial) = d_1 e^x\partial$ where $d_1 \in F^*$. By $\theta(e^{-x}\partial \ast e^x\partial) = \partial$, we have that $\theta(e^{-x}\partial) \ast d_1 e^x\partial = \partial$. This implies that

\[
(3) \quad \theta(e^{-x}\partial) = d_1^{-1} e^{-x}\partial.
\]

By $\theta(e^x\partial \ast e^x\partial) = e^{2x}\partial$, we have that

\[
(4) \quad \theta(e^{2x}\partial) = d_1^2 e^{2x}\partial.
\]

By (3) and (4), we may assume that $\theta(e^{kx}\partial) = d_1^k e^{kx}\partial$ holds by induction on $k \in \mathbb{N}$ of $e^{kx}\partial$. By $\theta(e^{x}\partial \ast e^{kx}\partial) = ke^{(k+1)x}\partial$, we have that $\theta(e^{(k+1)x}\partial) = d_1^{k+1} e^{(k+1)x}\partial$. This proves that $\theta(e^{kx}\partial) = d_1^k e^{kx}\partial$ holds for any $k \in \mathbb{N}$. Symmetrically, we can prove that

\[
(5) \quad \theta(e^{kx}\partial) = d_1^k e^{kx}\partial
\]

holds for any negative integer $k$ by (3). This implies that $\theta$ is the non-associative algebra automorphism $\theta_{+,-d_1}$ which is defined in Note 1.

**Case II.** Let us assume that $c = -1$ and $b_1 = -1$ hold. Let us put $\theta(\partial) = -\partial$ and $\theta(e^x\partial) = d_2 e^{-x}\partial$ where $d_2 \in F^*$. By $\theta(e^{x}\partial \ast e^x\partial) = e^{2x}\partial$, we have that

\[
(6) \quad \theta(e^{2x}\partial) = d_2^2 e^{-2x}\partial.
\]

By induction on $k \in \mathbb{N}$ of $e^{kx}\partial$, we can prove that

\[
(7) \quad \theta(e^{kx}\partial) = d_2^k e^{-kx}\partial.
\]

By $\theta(e^{-x}\partial \ast e^x\partial) = -\partial$, we have that $\theta(e^{-x}\partial) \ast d_2 e^{-x}\partial = -\partial$. This implies that $\theta(e^{-x}\partial) = d_2^{-1} e^x\partial$. By induction on $k \in \mathbb{N}$ of $e^{kx}\partial$, we can prove that

\[
(8) \quad \theta(e^{-kx}\partial) = d_2^{-k} e^{kx}\partial.
\]

This implies that $\theta$ is the non-associative algebra automorphism $\theta_{-,-d_2}$ which is defined in Note 1. By Case I and Case II, we have proven the lemma. $\square$
Theorem 3. The non-associative algebra automorphism group 

\[ \text{Aut}_{\text{non}}(W(1,0,0)) \]

of \( W(1,0,0) \) is generated by \( \theta_{+,d_1} \) and \( \theta_{-,d_2} \) which are defined in Note 1 where \( d_1, d_2 \in F^* \).

Proof. Let \( \theta \) be the non-associative algebra automorphism of \( W(1,0,0) \).

By Lemma 3, \( \theta \) is either \( \theta_{+,d_1} \) or \( \theta_{-,d_2} \) where \( d_1, d_2 \in F^* \). So \( \text{Aut}_{\text{non}}(W(1,0,0)) \) of \( W(1,0,0) \) is generated by \( \theta_{+,d_1} \) and \( \theta_{-,d_2} \). Therefore we have proven the theorem. \( \square \)

Corollary 3. The non-associative algebra automorphism group 

\[ \text{Aut}_{\text{non}}(W(1,0,0)) \]

of the non-associative algebra \( W(1,0,0) \) is a non-abelian group.

Proof. By Theorem 3, the non-associative algebra automorphism group 

\[ \text{Aut}_{\text{non}}(W(1,0,0)) \]

of the non-associative algebra \( W(1,0,0) \) is generated by \( \theta_{+,d_1} \) and \( \theta_{-,d_2} \) where \( d_1, d_2 \in F^* \). Thus it is enough to check that \( \theta_{+,d_1} \circ \theta_{-,d_2} \neq \theta_{-,d_2} \circ \theta_{+,d_1} \) where \( \circ \) is the composition of the non-associative algebra automorphisms \( \theta_{+,d_1} \) and \( \theta_{-,d_2} \). But it is trivial to check the inequality by taking some basis element of the non-associative algebra \( W(1,0,0) \). So let omit the remaining steps of its proof. \( \square \)

Proposition 2. The non-associative algebra \( W(1,0,0) \) is not isomorphic to the non-associative algebra \( W(0,1,0) \) as non-associative algebras.

Proof. Since the non-associative algebra \( W(0,1,0) \) has a right identity and the non-associative algebra \( W(1,0,0) \) does not have a right identity, the proof of the proposition is straightforward. So it is omitted. \( \square \)

Proposition 3. The Lie algebra \( W(n^+,0,n+s) \) is isomorphic to the Lie algebra \( W(0,n,n+s) \) as Lie algebras. The non-associative algebra 

\[ W(n^+,0,n+s) \]

is not isomorphic to the non-associative algebra \( W(0,n,n+s) \) as non-associative algebras.

Proof. It is standard to find isomorphisms between appropriate algebras, so the proof of the proposition is omitted. \( \square \)
Proposition 4. The Lie algebra $W(1,0,0)$ (resp. the non-associative algebra $\overline{W}(1,0,0)$) does not hold its Jacobian conjecture.

Proof. It is easy to define a non-zero endomorphism $\theta$ of $W(1,0,0)$ (resp. $\overline{W}(1,1,0)$) which is not surjective. This completes its proof. \qed

Proposition 3 shows that there are non-isomorphic two non-associative algebras whose corresponding Lie algebras (i.e., using the commutators of them) are isomorphic. This fact is one of the reasons to study non-associative algebras.

4. Derivations of $W(1,0,0)$

Note that the $F$-algebra $F[x,x^{-1}]$ is isomorphic to the $F$-algebra $F[e^{\pm x}]$ as $F$-algebras. Let $A$ be an $F$-algebra. An additive $F$-map $D$ from $A$ to itself is a derivation if $D(l_1 \ast l_2) = D(l_1) \ast l_2 + l_1 \ast D(l_2)$ for any $l_1, l_2 \in A$.

Note 2. For any basis element $e^{kx}\partial$ of the non-associative algebra $W(1,0,0)$, if we define an $F$-additive linear map $D_c$ of the non-associative algebra $W(1,0,0)$ as follows:

$$D_c(e^{kx}\partial) = cke^{kx}\partial$$

then $D_c$ can be linearly extended to a derivation of the non-associative algebra $W(1,0,0)$ where $c \in F$.

Lemma 4. For any derivation $D$ of the non-associative algebra $\overline{W}(1,0,0)$, if $D(\partial) = 0$, then $D$ is the derivation $D_c$ which is defined in Note 2.

Proof. Let $D$ be the derivation of the non-associative algebra $\overline{W}(1,0,0)$ in the lemma. Since $\partial$ is a left identity of $e^x\partial$, we have that $D(\partial) \ast e^x\partial + \partial \ast D(e^x\partial) = D(e^x\partial)$, i.e., $\partial \ast D(e^x\partial) = D(e^x\partial)$ by assumption. This implies that

$$D(e^x\partial) = ce^x\partial$$

for $c \in F$. We have two cases $c = 0$ or $c \neq 0$.

Case I. Let us assume that $c = 0$. By (9), we have that $D(e^x\partial) = 0$. By $D(e^{2x}\partial \ast e^x\partial) = D(e^{2x}\partial)$, we have that $D(e^x\partial) \ast e^x\partial + e^x\partial \ast D(e^x\partial) = D(e^{2x}\partial)$. This implies that $D(e^{2x}\partial) = 0$. By induction on $k \in N$ of $e^{kx}\partial$, we can prove that

$$D(e^{kx}\partial) = 0.$$  

For any $k \in N$, we have that $D(e^{-kx}\partial) \ast e^{(k+1)x}\partial = 0$ by (10). Since the left annihilator of $e^{(k+1)x}\partial$ is zero, this implies that $D(e^{kx}\partial) = 0$ holds for any negative integers. This implies that $D$ is the zero map of the non-associative algebra $\overline{W}(1,0,0)$. 

\[\]
Case II. Let us assume that $c \neq 0$. By (9), we have that $D(e^x \partial) = ce^x \partial$. By $D(e^x \partial * e^x \partial) = D(e^{2x} \partial)$, we also have that $ce^x \partial * e^x \partial + e^x \partial * ce^x \partial = D(e^{2x} \partial)$. This implies that $D(e^{2x} \partial) = 2ce^{2x} \partial$. By induction on $k \in \mathbb{N}$ of $e^{kx} \partial$, we can prove that
\begin{equation}
(11) \quad D(e^{kx} \partial) = kce^{kx} \partial.
\end{equation}
By $D(e^{-x} \partial * e^x \partial) = D(\partial)$, we have that $D(e^{-x} \partial) * e^x \partial + e^{-x} \partial * D(e^x \partial) = 0$. This implies that $D(e^{-x} \partial) = -ce^{-x} \partial$. Similarly to (11), by induction on $-k \in \mathbb{N}$ of $e^{kx} \partial$, we can also prove that
\begin{equation}
(12) \quad D(e^{kx} \partial) = kce^{kx} \partial.
\end{equation}
This implies that $D$ is the derivation $D_c$ in Note 2. Therefore we have proven the lemma. \qed

**Theorem 4.** For any derivation $D$ of the non-associative algebra $\overline{W(1, 0, 0)}$, $D = \sum_{c \in \mathbb{F}} D_c$ where $D_c$ is the derivation which is defined in Note 2.

**Proof.** Let $D$ be the derivation of the non-associative algebra $\overline{W(1, 0, 0)}$ in the theorem. Since $\partial$ annihilates itself, we have that $D(\partial) * \partial + \partial * D(\partial) = 0$. This implies that $D(\partial) = c_1 \partial$ for $c_1 \in \mathbb{F}$. It is easy to prove that $c_1 = 0$. So by Lemma 4, $D$ is $D_c$ for $c \in \mathbb{F}$. This implies that $D = \sum_{c \in \mathbb{F}} D_c$ where $D_c$ is the derivation which is defined in Note 2. Therefore we have proven Theorem 4. \qed

By Theorem 4, we know that every derivation the non-associative algebra $\overline{W(1, 0, 0)}$ is a scalar derivation. All the derivations of the non-associative algebras $\overline{W_{N, 0, 0, 1}}$, are found in the papers [1], [10], [11] and please refer to the definitions of the algebras $\overline{W_{N, 0, 0, 1}}$, and $\overline{W_{N, 0, 0, 1}}$, in the papers [2], [3]. Thus it is an interesting problem to find all the derivations of the non-associative algebras $\overline{W_{N, 0, 0, 1}}$. Also it is an interesting problem to find the non-associative algebra automorphism group $\text{Aut}_{\text{non}}(\overline{W_{N, 0, 0, 1}})$ of the non-associative algebras $\overline{W_{N, 0, 0, 1}}$.[1], [6], [9].

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