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Mukut Mani Tripathi and Jeong-Sik Kim

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MUKUT MANI TRIPATHI AND JEONG-SIK KIM

Abstract. $N(k)$-quasi Einstein manifolds are introduced and studied.

1. Introduction

A non-flat Riemannian manifold $(M^n, g)$ is said to be a quasi Einstein manifold \cite{2} if its Ricci tensor $S$ satisfies
\begin{equation}
S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \quad X, Y \in TM
\end{equation}
or equivalently, its Ricci operator $Q$ satisfies
\begin{equation}
Q = a I + b \eta \otimes \xi
\end{equation}
for some smooth functions $a$ and $b \neq 0$, where $\eta$ is a nonzero 1-form such that
\begin{equation}
g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1
\end{equation}
for the associated vector field $\xi$. The 1-form $\eta$ is called the associated 1-form and the unit vector field $\xi$ is called the generator of the manifold. In an $n$-dimensional quasi Einstein manifold the Ricci tensor has precisely two distinct eigenvalues $a$ and $a + b$, where $a$ is of multiplicity $(n - 1)$ and $a + b$ is simple \cite{2}. A proper $\eta$-Einstein contact metric manifold \cite{1, 5} is a natural example of a quasi Einstein manifold.

In this paper, we introduce the concept of $N(k)$-quasi Einstein manifolds. In section 2, it is proved that conformally flat quasi Einstein manifolds are certain $N(k)$-quasi Einstein manifolds. Semi-symmetric $N(k)$-quasi Einstein manifolds are studied in section 3. A necessary and a sufficient condition for an $N(k)$-quasi Einstein manifold to satisfy $R(\xi, X) \cdot S = 0$ are obtained in section 4. In section 5, Ricci-recurrent quasi Einstein manifolds are studied. In the last section, it is proved that a quasi-umbilical hypersurface of an $N(\bar{k})$-quasi Einstein manifold, such that it is normal to the generator of the ambient manifold, is an $N(k)$-quasi Einstein manifold.
2. \(N(k)\)-quasi Einstein manifolds

The \(k\)-nullity distribution \(N(k)\) \([7]\) of a Riemannian manifold \(M\) is defined by

\[
N(k) : p \rightarrow N_p(k) = \{ Z \in T_pM \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \}
\]

for all \(X, Y \in TM\), where \(k\) is some smooth function.

Motivated by the above definition, we give the following definition.

**Definition 2.1.** Let \((M^n, g)\) be a quasi Einstein manifold. If the generator \(\xi\) belongs to the \(k\)-nullity distribution \(N(k)\) for some smooth function \(k\), then we say that \((M^n, g)\) is an \(N(k)\)-quasi Einstein manifold.

Let \((M^n, g)\) be a quasi Einstein manifold. From (1.2) and (1.3), it follows that

\[
(2.1) \quad S(X, \xi) = (a + b) \eta(X),
\]

\[
(2.2) \quad Q\xi = (a + b) \xi,
\]

\[
(2.3) \quad r = na + b,
\]

where \(r\) is the scalar curvature of \(M^n\).

In an \(n\)-dimensional Riemannian manifold \((M^n, g)\), the conformal curvature tensor \(C\) is given by \([8]\)

\[
C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} \{ g(Y,Z) QX - g(X,Z) QY \}
+ S(Y,Z)X - S(X,Z)Y \]
\[
(2.4) \quad + \frac{r}{(n-1)(n-2)} \{ g(Y,Z)X - g(X,Z)Y \}.
\]

If \((M^n, g)\) is a conformally flat quasi Einstein manifold, then in view of (1.2), (2.3) and (2.4) we have

\[
R(X,Y)Z = \frac{(n-2)a - b}{n-1} \{ g(Y,Z)X - g(X,Z)Y \}
+ \frac{b}{n-2} \{ g(Y,Z) \eta(X)\xi - g(X,Z) \eta(Y)\xi \}
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \}
\]
\[
(2.5) \quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \}.
\]

Putting \(Z = \xi\), in the above equation, we obtain

\[
R(X,Y)\xi = \frac{a + b}{n-1} \{ \eta(Y)X - \eta(X)Y \},
\]

that is, in an \(n\)-dimensional conformally flat quasi Einstein manifold, the generator \(\xi\) belongs to the \(\frac{a + b}{n-1}\)-nullity distribution \(N\left(\frac{a + b}{n-1}\right)\). We can state this fact as the following:
Theorem 2.2. An $n$-dimensional conformally flat quasi Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi Einstein manifold.

Thus, we see that $n$-dimensional conformally flat quasi Einstein manifolds are natural examples of $N(k)$-quasi Einstein manifolds. It is well-known that in a 3-dimensional Riemannian manifold $(M^3, g)$, the conformal curvature tensor vanishes, therefore we have the following

Corollary 2.3. Each 3-dimensional quasi Einstein manifold is an $N\left(\frac{a+b}{2}\right)$-quasi Einstein manifold.

Let $(M^n, g)$ be an $N(k)$-quasi Einstein manifold. Then, we have

$$R(Y, Z)\xi = k(\eta(Z)Y - \eta(Y)Z).$$

The equation (2.7) is equivalent to

$$R(\xi, Y)Z = k(g(Y, Z)\xi - \eta(Z)Y).$$

In particular, the above equation implies that

$$R(\xi, Y)\xi = k(\eta(Y)\xi - Y).$$

From (2.7) and (2.8), we have

$$\eta(R(Y, Z)\xi) = 0,$$

$$\eta(R(\xi, Y)Z) = k(g(Y, Z) - \eta(Y)\eta(Z)).$$

3. Semi-symmetric $N(k)$-quasi Einstein manifolds

As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold $M$ is said to be semi-symmetric if its curvature tensor $R$ satisfies

$$R(X, Y)\cdot R = 0, \quad X, Y \in TM,$$

where $R(X, Y)$ acts on $R$ as a derivation. In this section, we study $N(k)$-quasi Einstein manifolds $M$ satisfying $R(\xi, X)\cdot R = 0, \quad X \in TM$.

First, we prove the following theorem.

Theorem 3.1. An $N(k)$-quasi Einstein manifold $(M^n, g)$ satisfies $R(\xi, X)\cdot R = 0$ if and only if $k = 0$.

Proof. The condition $R(\xi, X)\cdot R = 0$ implies that

$$0 = [R(\xi, X), R(Y, Z)]\xi - R(R(\xi, X)Y, Z)\xi - R(Y, R(\xi, X)Z)\xi,$$

which in view of (2.8) and (2.9) gives

$$0 = k \{g(X, R(Y, Z)\xi)\xi - \eta(X)R(Y, Z)\xi + R(Y, Z)X - g(X, Y)R(\xi, Z)\xi + \eta(Y)R(X, Z)\xi - g(X, Z)R(Y, \xi)\xi + \eta(Z)R(Y, X)\xi \}.$$
In view of (2.7), the above equation yields
\[ 0 = k \{ R(Y, Z) X - k (g(Z, X) Y - g(Y, X) Z) \}. \]
Therefore, either \( k = 0 \) or
\[ R(Y, Z) X = k (g(Z, X) Y - g(Y, X) Z). \]
In the second case, \( M^n \) becomes an Einstein space, which is not possible. Thus we have \( k = 0 \). Conversely, if \( k = 0 \), in view of (2.8) \( M^n \) satisfies \( R(\xi, X) \cdot R = 0 \). This completes the proof. \( \square \)

As a Corollary, we have the following

**Corollary 3.2.** If \( (M^n, g) \) is a semi-symmetric \( N(k)-\)quasi Einstein manifold, then \( k = 0 \).

Now, we apply the above two results to conformally flat quasi Einstein manifolds and in result we may state the following

**Theorem 3.3.** A conformally flat quasi Einstein manifold satisfies \( R(\xi, X) \cdot S = 0 \) if and only if \( a + b = 0 \). In particular, each conformally flat semi-symmetric quasi Einstein manifold satisfies \( a + b = 0 \).

4. **\( N(k) \)-quasi Einstein manifolds satisfying \( R(\xi, X) \cdot S = 0 \)**

First, we prove the following

**Theorem 4.1.** An \( N(k)-\)quasi Einstein manifold \( (M^n, g) \) satisfies \( R(\xi, X) \cdot S = 0 \) if and only if \( k = 0 \).

**Proof.** Let \( (M^n, g) \) be a \( N(k)-\)quasi Einstein manifold. The condition \( R(\xi, X) \cdot S = 0 \) gives
\[ (4.1) \quad S( R(\xi, X) Y, \xi) + S( Y, R(\xi, X) \xi) = 0. \]
In view of (2.1) and (2.11), we get
\[ (4.2) \quad S( R(\xi, X) Y, \xi) = (a + b) k (g(X, Y) - \eta(X) \eta(Y)). \]
In view of (2.9) and (2.1) we have
\[ (4.3) \quad S( R(\xi, X) \xi, Y) = -k S( X, Y) + (a + b) k \eta(X) \eta(Y). \]
From (4.1), (4.2) and (4.3), we have
\[ (4.4) \quad k \{ S - (a + b) g \} = 0. \]
Therefore, either \( k = 0 \) or \( S = (a + b) g \). In the second case, \( M^n \) becomes an Einstein space, which is not possible. Thus we have \( k = 0 \). Conversely, if \( k = 0 \) then in view of (2.8) \( M^n \) satisfies \( R(\xi, X) \cdot S = 0 \). \( \square \)

As an application, we have the following

**Theorem 4.2.** A conformally flat quasi Einstein manifold satisfies \( R(\xi, X) \cdot S = 0 \) if and only if \( a + b = 0 \).
5. Ricci-recurrent quasi Einstein manifolds

A non-flat Riemannian manifold \( M \) is called a \textit{Ricci-recurrent manifold} \cite{6} if its Ricci tensor \( S \) satisfies the condition
\[
(\nabla_X S)(Y, Z) = A(X) S(Y, Z),
\]
where \( \nabla \) is Levi-Civita connection of the Riemannian metric \( g \) and \( A \) is a 1-form on \( M \).

Now, we prove the following

\textbf{Theorem 5.1.} If \( M \) is a Ricci-recurrent quasi Einstein manifold, then
\[
(a + b) A(X) = X(a + b), \quad X \in TM.
\]

\textit{Proof.} Using (5.1) in
\[
(\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),
\]
we get
\[
A(X) S(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).
\]
Putting \( Y = Z = \xi \), in the above equation we obtain
\[
S(\xi, \xi) A(X) = XS(\xi, \xi) - 2S(\nabla_X \xi, \xi),
\]
from which, in view of (2.1), we get (5.2). \( \square \)

A Ricci-recurrent manifold is Ricci-symmetric if and only if the 1-form \( A \) is zero. Thus we have the following two corollaries:

\textbf{Corollary 5.2.} If \( M \) is a Ricci-symmetric quasi Einstein manifold, then \( a + b \) is constant.

\textbf{Corollary 5.3.} If \( M \) is a Ricci-recurrent quasi Einstein manifold and if \( a + b \) is constant, then either \( a + b = 0 \) or \( M \) reduces to a Ricci-symmetric quasi Einstein manifold.

6. Quasi-umbilical hypersurfaces

Let \( M^n \) be a hypersurface of a Riemannian manifold \((\bar{M}^{n+1}, g)\). We now assume that \( M^n \) is orientable and choose a unit vector field \( \xi \) of \( \bar{M}^{n+1} \) normal to \( M^n \). Then Gauss and Weingarten formulae are given respectively by

\[
(\bar{\nabla}_X Y) Y = \nabla_X Y + h(X, Y) \xi, \quad (X, Y \in TM^n),
\]
\[
\bar{\nabla}_X \xi = -H X,
\]
where \( \bar{\nabla} \) and \( \nabla \) are respectively the Riemannian and induced Riemannian connections in \( \bar{M}^{n+1} \) and \( M^n \) and \( h \) is the second fundamental form related to \( H \) by
\[
h(X, Y) = g(HX, Y).
\]
\(M^n\) is called a \textit{quasi-umbilical hypersurface} [3] if
\[(6.4) \quad h(X, Y) = \alpha g(X, Y) + \beta u(X) u(Y), \quad X, Y \in TM^n,\]
where \(\alpha\) and \(\beta\) are some smooth functions and \(u\) is a 1-form. A quasi-umbilical hypersurface becomes umbilical, geodesic or cylindrical according as \(\beta = 0, \alpha = 0 = \beta\) or \(\alpha = 0\).

If \(M^n\) is a quasi-umbilical hypersurface of a Riemannian manifold \((\bar{M}^{n+1}, g)\), then the Gauss equation becomes [4]
\[(6.5) \quad \bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + \alpha^2 (g(X, Z) g(Y, W) - g(X, W) g(Y, Z)) + \alpha \beta (u(Y) u(W) g(X, Z) + u(X) u(Z) g(Y, W) - u(Y) u(Z) g(X, W) - u(X) u(W) g(Y, Z))\]
for all \(X, Y, Z, W \in TM^n\). From the above equation, we get
\[(6.6) \quad \bar{S}(X, Y) = \bar{R}(\xi, X, Y, \xi) + S(X, Y) - (n\alpha^2 + \alpha \beta) g(X, Y) - (n - 1) \alpha \beta u(X) u(Y),\]
where \(\bar{S}\) and \(S\) are Ricci tensors of \(\bar{M}^{n+1}\) and \(M^n\) respectively.

Now, we prove the following:

\textbf{Theorem 6.1.} If \(M^n\) is a quasi-umbilical hypersurface of an \(N(k)\)-quasi Einstein manifold \((\bar{M}^{n+1}, g)\), such that \(M^n\) is normal to the generator \(\xi\) of \(\bar{M}^{n+1}\), then \(M^n\) is an \(N(k)\)-quasi Einstein manifold.

\textbf{Proof.} Using (2.8) in (6.6), we see that \(M^n\) is a quasi Einstein manifold. Moreover, using (2.7) in (6.5), we find that \(M^n\) is an \(N(k)\)-quasi Einstein manifold, where \(k = k + \alpha (\alpha + \beta)\). \(\square\)

\textbf{Remark 6.2.} The above result is also true in the following cases (a) if \(M^n\) is umbilical, geodesic or cylindrical and/or (b) \(\bar{M}^{n+1}\) is a conformally flat quasi Einstein manifold.

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\textbf{References}


Mukut Mani Tripathi  
Department of Mathematics and Astronomy  
Lucknow University  
Lucknow 226 007, India  
E-mail address: mtripathi66@yahoo.com

Jeong-Sik Kim  
Department of Mathematics and Mathematical Information  
Yosu National University  
Yosu 550-749, South Korea  
E-mail address: jskin315@hotmail.com