INTUITIONISTIC FUZZY SETS IN GAMMA-SEMIGROUPS

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INTUITIONISTIC FUZZY SETS IN GAMMA-SEMIGROUPS

MUSTAF A UÇKUN, MEHMET ALİ ÖZTÜRK, AND YOUNG BAE JUN

ABSTRACT. We consider the intuitionistic fuzzification of the concept of several Γ-ideals in a Γ-semigroup \( S \), and investigate some properties of such Γ-ideals.

1. Introduction

The notion of a fuzzy set in a set was introduced by L. A. Zadeh [10], and since then this concept has been applied to various algebraic structures. K. T. Atanassov [1] defined the notion of an intuitionistic fuzzy set, as a concept more general than a fuzzy set (see also [2]). Using fuzzy ideals, N. Kuroki [5] discussed characterizations of semigroups (see also [6]). K. H. Kim and Y. B. Jun [3] considered the intuitionistic fuzzification of the notion of several ideals in a semigroup, and investigated some properties of such ideals (see also [4]). M. K. Sen and N. K. Saha [9] defined the concept of a Γ-semigroup, and established a relation between regular Γ-semigroup and Γ-group (see also [7], [8]). In this paper, we introduce the notion of an intuitionistic fuzzy Γ-ideal of a Γ-semigroup, and we investigate some properties connected with intuitionistic fuzzy Γ-ideals in a Γ-semigroup.

2. Preliminaries

Let \( S = \{x, y, z, \ldots\} \) and \( \Gamma = \{\alpha, \beta, \gamma, \ldots\} \) be two non-empty sets. Then \( S \) is called a Γ-semigroup if it satisfies

- \( x\gamma y \in S \),
- \( x(\beta y)\gamma z = x\beta(y\gamma z) \)

for all \( x, y, z \in S \) and \( \beta, \gamma \in \Gamma \). A non-empty subset \( U \) of a Γ-semigroup \( S \) is said to be a Γ-subsemigroup of \( S \) if \( UTU \subseteq U \). A left (right) Γ-ideal of a Γ-semigroup \( S \) is a non-empty subset \( U \) of \( S \) such that \( STU \subseteq U \) (\( UTU \subseteq U \)). If \( U \) is both a left and a right Γ-ideal of a Γ-semigroup \( S \), then we say that \( U \) is

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a $\Gamma$-ideal of $S$. A $\Gamma$-subsemigroup $U$ of a $\Gamma$-semigroup $S$ is called an interior $\Gamma$-ideal of $S$ if $\text{UTSU} \subseteq U$. A $\Gamma$-bi-ideal of a $\Gamma$-semigroup $S$ is a $\Gamma$-subsemigroup $U$ of $S$ such that $\text{UTSTU} \subseteq U$. Let $L[x]$ denote the principal left $\Gamma$-ideal of a $\Gamma$-semigroup $S$ generated by $x$ in $S$, that is, $L[x] = \{x\} \cup S \Gamma x$. A $\Gamma$-semigroup $S$ is said to be regular if, for each $x \in S$, there exist $s \in S$ and $\beta, \gamma \in \Gamma$ such that $x = x s \beta x$. A $\Gamma$-semigroup $S$ is called left-zero (right-zero) if $x \gamma y = x$ ($x \gamma y = y$) for all $x, y \in S$ and $\gamma \in \Gamma$. A $\Gamma$-semigroup $S$ is said to be left (right) simple if $S$ has no proper left (right) $\Gamma$-ideals. If a $\Gamma$-semigroup $S$ has no proper $\Gamma$-ideals, then we say that $S$ is simple. An element $e$ in a $\Gamma$-semigroup $S$ is called an idempotent if $e \gamma e = e$ for some $\gamma \in \Gamma$. Let $E_S$ denote the set of all idempotents in a $\Gamma$-semigroup $S$.

By a fuzzy set $\mu$ in a non-empty set $X$ we mean a function $\mu : X \rightarrow [0, 1]$ and the complement of $\mu$, denoted by $\bar{\mu}$, is the fuzzy set in $X$ given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. An intuitionistic fuzzy set (briefly IFS) $A$ in a non-empty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set $A$, which is a subset of $X$, respectively, and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

for all $x \in X$. An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ in $X$ can be identified to an ordered pair $(\mu_A, \nu_A)$ in $I^X \times I^X$ where $I = [0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the IFS $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$. Let $\chi_U$ denote the characteristic function of a non-empty subset $U$ of a $\Gamma$-semigroup $S$.

3. Intuitionistic fuzzy $\Gamma$-ideals

In what follows, let $S$ denote a $\Gamma$-semigroup unless otherwise specified.

**Definition 3.1.** For an IFS $A = (\mu_A, \nu_A)$ in $S$, consider the following axioms:

$(\Gamma S_1)$ $\mu_A(x \gamma y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

$(\Gamma S_2)$ $\nu_A(x \gamma y) \leq \max\{\nu_A(x), \nu_A(y)\}$

for all $x, y \in S$ and $\gamma \in \Gamma$. Then $A = (\mu_A, \nu_A)$ is called a first (resp. second) intuitionistic fuzzy $\Gamma$-subsemigroup (briefly IFT$S_1$ (resp. IFT$S_2$)) of $S$ if it satisfies $(\Gamma S_1)$ (resp. $(\Gamma S_2)$). Also, $A = (\mu_A, \nu_A)$ is said to be an intuitionistic fuzzy $\Gamma$-subsemigroup (briefly IFT$S$) of $S$ if it is both a first and a second intuitionistic fuzzy $\Gamma$-subsemigroup.

**Theorem 3.2.** If $U$ is a $\Gamma$-subsemigroup of $S$, then $\bar{U} = (\chi_U, \bar{\chi}_U)$ is an IFT$S$ of $S$. 
Proof. Let \( x, y \in S \) and \( \gamma \in \Gamma \). From the hypothesis, \( x\gamma y \in U \) if \( x, y \in U \). In this case
\[
\chi_U(x\gamma y) = 1 \geq \min\{\chi_U(x), \chi_U(y)\}
\]
and
\[
\bar{\chi}_U(x\gamma y) = 1 - \chi_U(x\gamma y) \\
\leq 1 - \min\{\chi_U(x), \chi_U(y)\} \\
= \max\{1 - \chi_U(x), 1 - \chi_U(y)\} \\
= \max\{\bar{\chi}_U(x), \bar{\chi}_U(y)\}.
\]
If \( x \notin U \) or \( y \notin U \), then \( \chi_U(x) = 0 \) or \( \chi_U(y) = 0 \). Thus
\[
\chi_U(x\gamma y) \geq 0 = \min\{\chi_U(x), \chi_U(y)\}
\]
and
\[
\max\{\chi_U(x), \chi_U(y)\} = \max\{1 - \chi_U(x), 1 - \chi_U(y)\} \\
= 1 - \min\{\chi_U(x), \chi_U(y)\} \\
= 1 \geq \bar{\chi}_U(x\gamma y).
\]
This completes the proof. \( \square \)

**Theorem 3.3.** Let \( U \) be a non-empty subset of \( S \). If \( \bar{U} = (\chi_U, \bar{\chi}_U) \) is an IFTS\(_1\) or IFTS\(_2\) of \( S \), then \( U \) is a \( \Gamma \)-subsemigroup of \( S \).

Proof. Suppose that \( \bar{U} = (\chi_U, \bar{\chi}_U) \) is an IFTS\(_1\) of \( S \) and \( x \in \overline{U} U \). In this case, \( x = u\gamma v \) for some \( u, v \in U \) and \( \gamma \in \Gamma \). It follows from (\( \Gamma \)S\(_1\)) that
\[
\chi_U(x) = \chi_U(u\gamma v) \geq \min\{\chi_U(u), \chi_U(v)\} = 1.
\]
Hence \( \chi_U(x) = 1 \), i.e. \( x \in U \). Thus \( U \) is a \( \Gamma \)-subsemigroup of \( S \).

Now, assume that \( \bar{U} = (\chi_U, \bar{\chi}_U) \) is an IFTS\(_2\) of \( S \) and \( x' \in \overline{U} U \). Then \( x' = u'\gamma' v' \) for some \( u', v' \in U \) and \( \gamma' \in \Gamma \). Using (\( \Gamma \)S\(_2\)), we get that
\[
\bar{\chi}_U(x') = \bar{\chi}_U(u'\gamma' v') \\
\leq \max\{\bar{\chi}_U(u'), \bar{\chi}_U(v')\} \\
= \max\{1 - \chi_U(u'), 1 - \chi_U(v')\} = 0
\]
and so \( \bar{\chi}_U(x') = 1 - \chi_U(x') = 0 \). Therefore \( \chi_U(x') = 1 \), i.e. \( x' \in U \). This completes the proof. \( \square \)

**Definition 3.4.** For an IFS \( A = (\mu_A, \nu_A) \) in \( S \), consider the following axioms:
(\( \Gamma \)I\(_1\)) \( \mu_A(x\gamma y) \geq \mu_A(y) \),
(\( \Gamma \)I\(_2\)) \( \nu_A(x\gamma y) \leq \nu_A(y) \)
for all \( x, y \in S \) and \( \gamma \in \Gamma \). Then \( A = (\mu_A, \nu_A) \) is called a first (resp. second) intuitionistic fuzzy left \( \Gamma \)-ideal (briefly IFLI\(_1\) (resp. IFLI\(_2\))) of \( S \) if it satisfies (\( \Gamma \)I\(_1\)) (resp. (\( \Gamma \)I\(_2\))). Also, \( A = (\mu_A, \nu_A) \) is said to be an intuitionistic fuzzy left \( \Gamma \)-ideal (briefly IFLI) of \( S \) if it is both a first and a second intuitionistic fuzzy left \( \Gamma \)-ideal.
Definition 3.5. For an IFS $A = (\mu_A, \nu_A)$ in $S$, consider the following axioms:

(RGI1) $\mu_A(x \gamma y) \geq \mu_A(x)$,
(RGI2) $\nu_A(x \gamma y) \leq \nu_A(x)$

for all $x, y \in S$ and $\gamma \in \Gamma$. Then $A = (\mu_A, \nu_A)$ is called a first (resp. second) intuitionistic fuzzy right $\Gamma$-ideal (briefly IFRI1 (resp. IFRI2)) of $S$ if it satisfies (RGI1) (resp. (RGI2)). Also, $A = (\mu_A, \nu_A)$ is said to be an intuitionistic fuzzy right $\Gamma$-ideal (briefly IFRI) of $S$ if it is both a first and a second intuitionistic fuzzy right $\Gamma$-ideal.

Definition 3.6. Let $A = (\mu_A, \nu_A)$ be an IFS in $S$. Then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy $\Gamma$-ideal (briefly IFI) of $S$ if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right $\Gamma$-ideal.

Proposition 3.7. Let $U$ be a left-zero $\Gamma$-subsemigroup of $S$. If $A = (\mu_A, \nu_A)$ is an IFLGI of $S$, then the restriction of $A$ to $U$ is constant, that is, $A(x) = A(y)$ for all $x, y \in U$.

Proof. Let $x, y \in U$. Since $U$ is left-zero, $x \gamma y = x$ and $y \gamma x = y$ for all $\gamma \in \Gamma$. In this case, from the hypothesis, we have that

$$\mu_A(x) = \mu_A(x \gamma y) \geq \mu_A(y),$$
$$\mu_A(y) = \mu_A(y \gamma x) \geq \mu_A(x)$$

and

$$\nu_A(x) = \nu_A(x \gamma y) \leq \nu_A(y),$$
$$\nu_A(y) = \nu_A(y \gamma x) \leq \nu_A(x).$$

Thus we obtain $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$ for all $x, y \in U$. Hence $A(x) = A(y)$ for all $x, y \in U$. □

Lemma 3.8. If $U$ is a left $\Gamma$-ideal of $S$, then $\hat{U} = (\chi_U, \bar{\chi}_U)$ is an IFLGI of $S$.

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. Since $U$ is a left $\Gamma$-ideal of $S$, $x \gamma y \in U$ if $y \in U$.

It follows that

$$\chi_U(x \gamma y) = 1 = \chi_U(y)$$

and

$$\bar{\chi}_U(x \gamma y) = 1 - \chi_U(x \gamma y) = 0 = 1 - \chi_U(y) = \bar{\chi}_U(y).$$

If $y \notin U$, then $\chi_U(y) = 0$. In this case

$$\chi_U(x \gamma y) \geq 0 = \chi_U(y)$$

and

$$\bar{\chi}_U(y) = 1 - \chi_U(y) = 1 \geq \bar{\chi}_U(x \gamma y).$$

Consequently, $\hat{U} = (\chi_U, \bar{\chi}_U)$ is an IFLGI of $S$. □

Theorem 3.9. Let $A = (\mu_A, \nu_A)$ be an IFLGI of $S$. If $E_S$ is a left-zero $\Gamma$-subsemigroup of $S$, then $A(e) = A(e')$ for all $e, e' \in E_S$. 
Proof. Let $e, e' \in E_S$. From the hypothesis, $e \gamma e' = e$ and $e' \gamma e = e'$ for all $\gamma \in \Gamma$. Thus, since $A = (\mu_A, \nu_A)$ is an IFLTI of $S$, we get that
\[
\mu_A(e) = \mu_A(e \gamma e') \geq \mu_A(e'),
\]
\[
\mu_A(e') = \mu_A(e' \gamma e) \geq \mu_A(e)
\]
and
\[
\nu_A(e) = \nu_A(e \gamma e') \leq \nu_A(e'),
\]
\[
\nu_A(e') = \nu_A(e' \gamma e) \leq \nu_A(e).
\]
Hence we have $\mu_A(e) = \mu_A(e')$ and $\nu_A(e) = \nu_A(e')$ for all $e, e' \in E_S$. This completes the proof. $\square$

**Theorem 3.10.** Let $S$ be regular. If, for every non-empty subset $U$ of $S$, $U = (\chi_U, \bar{\chi}_U)$ is an IFLTI$_1$ (or IFLTI$_2$) of $S$ and $\bar{U}(e) = U(e')$ for all $e, e' \in E_S$, then $E_S$ is a left-zero $\Gamma$-subsemigroup of $S$.

**Proof.** Since $S$ is regular, $E_S$ is non-empty. Let $e = e \gamma e', e' = e' \gamma' e' \in E_S$ where $\gamma, \gamma' \in \Gamma$. Because of $S$ is regular, $L[e] = S \Gamma e$. Since $L[e]$ is a left $\Gamma$-ideal of $S$, we obtain $\bar{L}[e] = (\chi_{L[e]}, \bar{\chi}_{L[e]})$ is an IFLTI$_1$ (or IFLTI$_2$) of $S$ by Lemma 3.8. In this case, from the hypothesis, we get that
\[
\chi_{L[e]}(e') = \chi_{L[e]}(e) = 1 \quad (or \quad \bar{\chi}_{L[e]}(e') = \bar{\chi}_{L[e]}(e) = 0).
\]
Hence $e' \in L[e] = S \Gamma e$. Thus
\[
e' = x \beta e = x \beta (e \gamma e) = (x \beta e) \gamma e = e' \gamma e
\]
for some $x \in S$ and $\beta \in \Gamma$. Consequently, $E_S$ is a left-zero $\Gamma$-semigroup. $\square$

**Definition 3.11.** For an IFS $A = (\mu_A, \nu_A)$ in $S$, consider the following axioms:

(IITI$_1$) $\mu_A(x; s \beta \gamma y) \geq \mu_A(s)$,

(IITI$_2$) $\nu_A(x; s \beta \gamma y) \leq \nu_A(s)$

for all $s, x, y \in S$ and $\beta, \gamma \in \Gamma$. Then $A = (\mu_A, \nu_A)$ is called a first (resp. second) intuitionistic fuzzy interior $\Gamma$-ideal (briefly IITI$_1$ (resp. IITI$_2$)) of $S$ if it is an IFTSI$_1$ (resp. IFTSI$_2$) satisfying (IITI$_1$) (resp. (IITI$_2$)). Also, $A = (\mu_A, \nu_A)$ is said to be an intuitionistic fuzzy interior $\Gamma$-ideal (briefly IITI) of $S$ if it is both a first and a second intuitionistic fuzzy interior $\Gamma$-ideal.

**Remark 3.12.** It is clear that every IFTI of $S$ is an IITI of $S$.

**Theorem 3.13.** If $S$ is regular, then every IITI of $S$ is an IFTI of $S$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an IITI of $S$ and $x, y \in S$. In this case, because of $S$ is regular, there exist $s, s' \in S$ and $\beta, \beta', \gamma, \gamma' \in \Gamma$ such that $x = x \beta s \gamma x$
and \( y = y^\beta s^\gamma y \). Thus
\[
\mu_A(x \alpha^\beta y) = \mu_A(x \alpha^\beta(y^\beta s^\gamma y)) \\
= \mu_A(x \alpha^\beta y^\beta(s^\gamma y)) \\
\geq \mu_A(y)
\]
and
\[
\nu_A(x \alpha^\beta y) = \nu_A(x \alpha^\beta(y^\beta s^\gamma y)) \\
= \nu_A(x \alpha^\beta y^\beta(s^\gamma y)) \\
\leq \nu_A(y)
\]
for all \( \alpha^\prime \in \Gamma \). It follows that \( A = (\mu_A, \nu_A) \) is an \( IFLI_2 \) of \( S \). Similarly, we can show that \( A = (\mu_A, \nu_A) \) is an \( IFI_1 \) of \( S \). This completes the proof.

**Theorem 3.14.** If \( U \) is an interior \( \Gamma \)-ideal of \( S \), then \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFLI_2 \) of \( S \).

**Proof.** Since \( U \) is a \( \Gamma \)-subsemigroup of \( S \), we have that \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFI_2 \) of \( S \) by Theorem 3.2. Let \( s, x, y \in S \) and \( \beta, \gamma \in \Gamma \). From the hypothesis, \( x^\beta s^\gamma y \in U \) if \( s \in U \). In this case
\[
\chi_U(x^\beta s^\gamma y) = 1 = \chi_U(s)
\]
and
\[
\bar{\chi}_U(x^\beta s^\gamma y) = 1 - \chi_U(x^\beta s^\gamma y) = 0 = 1 - \chi_U(s) = \bar{\chi}_U(s).
\]
If \( s \notin U \), then \( \chi_U(s) = 0 \). Thus
\[
\chi_U(x^\beta s^\gamma y) \geq 0 = \chi_U(s)
\]
and
\[
\bar{\chi}_U(s) = 1 - \chi_U(s) = 1 \geq \bar{\chi}_U(x^\beta s^\gamma y).
\]
Consequently, \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFLI_2 \) of \( S \). □

**Theorem 3.15.** Let \( S \) be regular and \( U \) a non-empty subset of \( S \). If \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFI_1 \) or \( IFI_2 \) of \( S \), then \( U \) is an interior \( \Gamma \)-ideal of \( S \).

**Proof.** It is clear that \( U \) is a \( \Gamma \)-subsemigroup of \( S \) by Theorem 3.3. Suppose that \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFI_1 \) of \( S \) and \( x \in STUS \). In this case, \( x = s^\beta u^\gamma t \) for some \( s, t \in S, u \in U \) and \( \beta, \gamma \in \Gamma \). It follows from \( (II_1) \) that
\[
\chi_U(x) = \chi_U(s^\beta u^\gamma t) \geq \chi_U(u) = 1.
\]
Hence \( \chi_U(x) = 1 \), i.e. \( x \in U \). Thus \( U \) is an interior \( \Gamma \)-ideal of \( S \).

Now, assume that \( \tilde{U} = (\chi_U, \bar{\chi}_U) \) is an \( IFI_2 \) of \( S \) and \( x' \in STUS \). Then \( x' = s'^\beta u'^\gamma t' \) for some \( s', t' \in S, u' \in U \) and \( \beta', \gamma' \in \Gamma \). Using \( (II_2) \), we obtain
\[
\bar{\chi}_U(x') = \bar{\chi}_U(s'^\beta u'^\gamma t') \leq \bar{\chi}_U(u') = 1 - \chi_U(u') = 0
\]
and so \( \chi_U(x') = 1 - \chi_U(x') = 0 \). Therefore \( \chi_U(x') = 1 \), i.e. \( x' \in U \). This completes the proof. □
Definition 3.16. S is called first (resp. second) intuitionistic fuzzy left simple if every \(IFL_{I_1}\) (resp. \(IFL_{I_2}\)) of S is constant. Also, S is said to be intuitionistic fuzzy left simple if it is both first and second intuitionistic fuzzy left simple, i.e. every \(IFL_{I}\) of S is constant.

Theorem 3.17. If S is left simple, then S is intuitionistic fuzzy left simple.

Proof. Let \(A = (\mu_A, \nu_A)\) be an \(IFL_{I}\) of S and \(x, x' \in S\). In this case, because of S is left simple, there exist \(s, s' \in S\) and \(\gamma, \gamma' \in \Gamma\) such that \(x = s\gamma x'\) and \(x' = s'\gamma' x\). Thus, since \(A = (\mu_A, \nu_A)\) is an \(IFL_{I}\) of S, we get that

\[
\mu_A(x) = \mu_A(s\gamma x') \geq \mu_A(x'),
\]

\[
\mu_A(x') = \mu_A(s'\gamma' x) \geq \mu_A(x)
\]

and

\[
\nu_A(x) = \nu_A(s\gamma x') \leq \nu_A(x'),
\]

\[
\nu_A(x') = \nu_A(s'\gamma' x) \leq \nu_A(x).
\]

Hence we have \(\mu_A(x) = \mu_A(x')\) and \(\nu_A(x) = \nu_A(x')\) for all \(x, x' \in S\), that is, \(A(x) = A(x')\) for all \(x, x' \in S\). Consequently, S is intuitionistic fuzzy left simple. \(\square\)

Theorem 3.18. If S is first or second intuitionistic fuzzy left simple, then S is left simple.

Proof. Let \(U\) be a left \(\Gamma\)-ideal of S. Suppose that S is first (or second) intuitionistic fuzzy left simple. Because of \(\tilde{U} = (\chi_U, \bar{\chi}_U)\) is an \(IFL_{I}\) of S by Lemma 3.8, \(\tilde{U} = (\chi_U, \bar{\chi}_U)\) is an \(IFL_{I}\) (and \(IFL_{I_2}\)) of S. From the hypothesis, \(\chi_U\) (and \(\bar{\chi}_U\)) is constant. Since U is non-empty, it follows that \(\chi_U = 1\) (or \(\bar{\chi}_U = 0\)), where \(1\) and \(0\) are fuzzy sets in S defined by \(1(x) = 1\) and \(0(x) = 0\) for all \(x \in S\), respectively. Thus \(x \in U\) for all \(x \in S\). This completes the proof. \(\square\)

Theorem 3.19. If S is simple, then every \(IFL_{I}\) of S is constant.

Proof. Let \(A = (\mu_A, \nu_A)\) be an \(IFL_{I}\) of S and \(x, x' \in S\). In this case, because of S is simple, there exist \(s, s', t, t' \in S\) and \(\beta, \beta', \gamma, \gamma' \in \Gamma\) such that \(x = s\beta x'\gamma t\) and \(x' = s'\beta' x'\gamma' t'\). Thus, since \(A = (\mu_A, \nu_A)\) is an \(IFL_{I}\) of S, we obtain that

\[
\mu_A(x) = \mu_A(s\beta x'\gamma t) \geq \mu_A(x'),
\]

\[
\mu_A(x') = \mu_A(s'\beta' x'\gamma' t') \geq \mu_A(x)
\]

and

\[
\nu_A(x) = \nu_A(s\beta x'\gamma t) \leq \nu_A(x'),
\]

\[
\nu_A(x') = \nu_A(s'\beta' x'\gamma' t') \leq \nu_A(x).
\]

Hence we get \(\mu_A(x) = \mu_A(x')\) and \(\nu_A(x) = \nu_A(x')\) for all \(x, x' \in S\). Consequently, \(A = (\mu_A, \nu_A)\) is constant. \(\square\)
Definition 3.20. For an IFTS $A = (\mu_A, \nu_A)$ in $S$, consider the following axioms:

\( (\Gamma B_1) \quad \mu_A(x\beta s\gamma y) \geq \min\{\mu_A(x), \mu_A(y)\} \),
\( (\Gamma B_2) \quad \nu_A(x\beta s\gamma y) \leq \max\{\nu_A(x), \nu_A(y)\} \)

for all $s, x, y \in S$ and $\beta, \gamma \in \Gamma$. Then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy $\Gamma$-bi-ideal (briefly IFTB) of $S$ if it satisfies ($\Gamma B_1$) and ($\Gamma B_2$).

Remark 3.21. It is clear that every IFTI of $S$ is an IFTB of $S$.

Theorem 3.22. If $S$ is left simple, then every IFTB of $S$ is an IFRTI of $S$.

Proof. Let $A = (\mu_A, \nu_A)$ be an IFTB of $S$ and $x, y \in S$. In this case, from the hypothesis, there exist $s \in S$ and $\gamma \in \Gamma$ such that $y = s\gamma x$. Thus, because of $A = (\mu_A, \nu_A)$ is an IFTB of $S$, we have that

\[ \mu_A(x\beta y) = \mu_A(x\beta s\gamma x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x) \]

and

\[ \nu_A(x\beta y) = \nu_A(x\beta s\gamma x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x) \]

for all $\beta \in \Gamma$. It follows that $A = (\mu_A, \nu_A)$ is an IFRTI of $S$. \qed

Proposition 3.23. If $U$ is a $\Gamma$-bi-ideal of $S$, then $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an IFTB of $S$.

Proof. Since $U$ is a $\Gamma$-subsemigroup of $S$, we obtain that $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an IFTS of $S$ by Theorem 3.2. Let $s, x, y \in S$ and $\beta, \gamma \in \Gamma$. From the hypothesis, $x\beta s\gamma y \in U$ if $x, y \in U$. In this case

\[ \chi_U(x\beta s\gamma y) = 1 = \min\{\chi_U(x), \chi_U(y)\} \]

and

\[ \bar{\chi}_U(x\beta s\gamma y) = 1 - \chi_U(x\beta s\gamma y) = 0 = \max\{\bar{\chi}_U(x), \bar{\chi}_U(y)\} \].

If $x \notin U$ or $y \notin U$, then $\chi_U(x) = 0$ or $\chi_U(y) = 0$. Thus

\[ \chi_U(x\beta s\gamma y) \geq 0 = \min\{\chi_U(x), \chi_U(y)\} \]

and

\[ \max\{\bar{\chi}_U(x), \bar{\chi}_U(y)\} = \max\{1 - \chi_U(x), 1 - \chi_U(y)\} \]
\[ = 1 - \min\{\chi_U(x), \chi_U(y)\} \]
\[ = 1 \geq \bar{\chi}_U(x\beta s\gamma y). \]

Consequently, $\tilde{U} = (\chi_U, \bar{\chi}_U)$ is an IFTB of $S$. \qed
References


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