SEMIGROUPS OF TRANSFORMATIONS
WITH INVARIANT SET

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Abstract. Let $T(X)$ denote the semigroup (under composition) of transformations from $X$ into itself. For a fixed nonempty subset $Y$ of $X$, let $S(X,Y) = \{ \alpha \in T(X) : Y \alpha \subseteq Y \}$. Then $S(X,Y)$ is a semigroup of total transformations of $X$ which leave a subset $Y$ of $X$ invariant. In this paper, we characterize when $S(X,Y)$ is isomorphic to $T(Z)$ for some set $Z$ and prove that every semigroup $A$ can be embedded in $S(A^1,A)$. Then we describe Green’s relations for $S(X,Y)$ and apply these results to obtain its group $H$-classes and ideals.

1. Introduction

The full transformation semigroup $T(X)$ is extremely important and the Green’s relations play an essential role in semigroup theory. As far back in 1952, Malcev [5] determined ideals of $T(X)$, later in 1955 Miller and Doss [2] described its Green’s relations and group $H$-classes. This paper is devoted to generalizations of these results.

The semigroup we consider is $S(X,Y)$ consists of all mappings in $T(X)$ which leave $Y \subseteq X$ invariant. To the extent that $S(X,X) = T(X)$, we may regard $S(X,Y)$ as a generalization of $T(X)$.

Magill [4] introduced and studied the semigroup $S(X,Y)$ in 1966. Later in 1975, Symons [7] described the automorphism group of this semigroup. In 2005 Nenthein, Youngkhong, and Kemprasit [6] showed that $S(X,Y)$ is a regular semigroup if and only if $X = Y$ or $Y$ contains exactly one element, and $E = \{ \alpha \in S(X,Y) : X\alpha \cap Y = Y\alpha \}$ is the set of all regular elements of $S(X,Y)$. Here, in Section 2, we prove that: $S(X,Y)$ is isomorphic to $T(Z)$ if and only if $X = Y$ and $|Y| = |Z|$, and we also prove that every semigroup $A$ can be embedded in $S(A^1,A)$. In Section 3, we characterize Green’s relations on $S(X,Y)$ and find that $D = J$ if and only if $X$ is a finite set or $X = Y$ or $|Y| = 1$. 

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and we also show that its group $\mathcal{H}$-class is isomorphic to a certain subgroup of a permutation group. In Section 4, we describe ideals of the semigroup $S(X,Y)$.

Throughout the paper, the set $X$ we consider can be finite or infinite. The cardinality of a set $A$ is denoted by $|A|$ and $X = A \cup B$ means $X$ is a disjoint union of $A$ and $B$. Also, we write functions on the right; in particular, this means that for a composition $\alpha \beta$, $\alpha$ is applied first.

2. Isomorphisms and embeddings

Let $X$ be any set and $Y$ a fixed nonempty subset of $X$. We consider the subsemigroup of $T(X)$ defined by

$$S(X,Y) = \{ \alpha \in T(X) : Y \alpha \subseteq Y \},$$

where $Y \alpha$ denotes the range of $Y$ under $\alpha$. Note that $id_X$, the identity map on $X$, belongs to $S(X,Y)$ and

$$E = \{ \alpha \in S(X,Y) : X \alpha \cap Y = Y \alpha \}$$

is the set of all regular elements of $S(X,Y)$.

As in Clifford and Preston [1] vol 2, p. 241, we shall use the notation

$$\alpha = \left( \begin{array}{c} X_i \\ a_i \end{array} \right)$$

to mean that $\alpha \in T(X)$ and take as understood that the subscript $i$ belongs to some (unmentioned) index set $I$, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i \alpha^{-1} = X_i$.

With the above notation, for any $\alpha \in S(X,Y)$ we can write

$$\alpha = \left( \begin{array}{ccc} A_i & B_j & C_k \\ a_i & b_j & c_k \end{array} \right),$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X \setminus Y; \{a_i\} \subseteq Y, \{b_j\} \subseteq Y \setminus \{a_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Here, $I$ is a nonempty set, but $J$ or $K$ can be empty. For examples: If $\alpha \in E$, then $J$ is an empty set. And if $\alpha \in S(X,Y) \setminus E$, then both $I$ and $J$ are nonempty but $K$ can be an empty set.

The following example shows that in general $E$ is not a subsemigroup of $S(X,Y)$.

Example 1. (a) Let $X = \{1,2,3\}$ and $Y = \{1,2\}$. Define

$$\alpha = \left( \begin{array}{c} \{1,2\} \\ 1 \end{array} \right), \beta = \left( \begin{array}{c} \{2,3\} \\ 1 \end{array} \right).$$

Then we have $\alpha, \beta \in E$, but

$$\alpha \beta = \left( \begin{array}{c} \{1,2\} \\ 1 \end{array} \right) \notin E.$$
(b) Let $X = \mathbb{N}$ denote the set of positive integers and $Y = \{1, 2\}$. Define

$$\alpha = \begin{pmatrix} [1, 2] & X \setminus \{1, 2\} \\ 1 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & X \setminus \{1\} \\ 1 & 2 \end{pmatrix}. $$

Thus we have $\alpha, \beta \in E$, but

$$\alpha \beta = \begin{pmatrix} [1, 2] & X \setminus \{1, 2\} \\ 1 & 2 \end{pmatrix} \notin E. $$

Therefore, $E$ in (a) and (b) are not subsemigroups of $S(X, Y)$.

To give a necessary and sufficient condition for $E$ to be a regular subsemigroup, we first note the following.

1. If $X = Y$, then $E = S(X, Y) = T(X)$ which is a regular semigroup.
2. If $|Y| = 1$, say $Y = \{a\}$, then for each $\alpha \in S(X, Y)$ we have $X\alpha \cap Y = \{a\} = Y\alpha$, so $S(X, Y) = E$.

**Lemma 1.** The following statements are equivalent:

1. $E$ is a regular subsemigroup of $S(X, Y)$.
2. $S(X, Y)$ is a regular semigroup.
3. $X = Y$ or $|Y| = 1$.

**Proof.** From [6], Corollary 2.4, we have $S(X, Y)$ is regular if and only if $X = Y$ or $|Y| = 1$. Now, assume that $E$ is a regular subsemigroup of $S(X, Y)$ and suppose that $Y \not\subseteq X$ and $|Y| \geq 2$. Let $a, b \in Y$ be such that $a \neq b$ and $c \in X \setminus Y$. Define $\alpha, \beta \in E$ by

$$\alpha = \begin{pmatrix} Y & \setminus Y \\ a & c \end{pmatrix}, \quad \beta = \begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix}. $$

Then $\alpha \beta = \begin{pmatrix} Y & \setminus Y \\ a & b \end{pmatrix} \notin E$ which is a contradiction. Therefore, $Y = X$ or $|Y| = 1$. Conversely, assume that $X = Y$ or $|Y| = 1$. If $X = Y$, then $E = T(X)$ which is a regular semigroup. If $|Y| = 1$, then $S(X, Y)$ is regular and $E = S(X, Y)$, thus $E$ is a regular subsemigroup. \qed

If $X$ is an infinite set and $Y$ is a finite subset of $X$, then $X \neq Y$. Thus we have the following corollary.

**Corollary 1.** If $X$ is an infinite set and $Y$ is a finite subset of $X$, then $E = S(X, Y)$ is regular if and only if $|Y| = 1$.

**Theorem 1.** $S(X, Y) \cong T(Z)$ for some set $Z$ if and only if $X = Y$ and $|Y| = |Z|$.\[\]

**Proof.** If $X = Y$ and $|Y| = |Z|$, then $S(X, Y) = S(Y, Y) = T(Y) \cong T(Z)$. Conversely, assume that $S(X, Y) \cong T(Z)$. Suppose that $Y \not\subseteq X$, then $|X| > 1$. If $|Y| = 1$, say $Y = \{a\}$, then $S(X, Y)$ is a regular semigroup (by Lemma 1) with more than one element and having

$$\alpha = \begin{pmatrix} X \\ a \end{pmatrix} $$
as a zero element. That means \( S(X, Y) \cong T(Z) \) is a semigroup with zero which is a contradiction. But, if \(|Y| > 1\), then \( S(X, Y) \) is not regular, thus \( S(X, Y) \not\cong T(Z) \) which is a contradiction. Therefore \( X = Y \) and hence \( S(X, Y) = S(Y, Y) = T(Y) \). Thus \( T(Y) \cong T(Z) \) and this gives \(|Y| = |Z|\).  \(\square\)

Since \( X \neq Y \) when \( X \) is an infinite set and \( Y \) is a finite subset of \( X \), the following corollary is an immediate consequence of Theorem 1.

**Corollary 2.** If \( X \) is an infinite set and \( Y \) is a finite subset of \( X \), then \( S(X, Y) \) is never isomorphic to \( T(Z) \) for any set \( Z \).

From Theorem 1, we see that \( S(X, Y) \) is almost never isomorphic to \( T(Z) \). However, we can embedded \( T(Y) \) into \( S(X, Y) \) by sending \( \alpha \mapsto \alpha' \) where \( \alpha' \in S(X, Y) \) is defined by

\[
xa' = \begin{cases} 
  x\alpha & \text{if } x \in Y, \\
  x & \text{if } x \in X \setminus Y.
\end{cases}
\]

In 1959, M. Hall ([3], Theorem 1.1.2) showed that every semigroup \( A \) can be embedded in the full transformation semigroup by using the extended right regular representation of \( A \). That is for each \( a \in A \), define a map \( \rho_a : A^1 \to A^1 \) by \( x\rho_a = xa \) \((x \in A^1)\). Then \( \rho_a \in T(A^1) \) and \( \Phi : A \to T(A^1) \) given by \( a\Phi = \rho_a \) is a monomorphism. Since for each \( a \in A \), we have \( x\rho_a = xa \in A \) for all \( x \in A \), it follows that \( \rho_a \in S(A^1, A) \) and so \( \Phi \) maps \( A \) into \( S(A^1, A) \) is a well-defined monomorphism. That means \( A \) can be embedded in \( S(A^1, A) \) which is a proper non-regular subsemigroup of \( T(A^1) \) if \( A \) does not contains an identity element. Thus we have proved the following theorem.

**Theorem 2.** Every semigroup \( A \) can be embedded in \( S(A^1, A) \).

### 3. Green’s relations on \( S(X, Y) \)

Since \( S(X, Y) \) is not a regular subsemigroup of \( T(X) \) if \( Y \subset X \) and \(|Y| > 1\), Hall’s theorem ([3], Proposition 2.4.2) cannot be applied to find the \( \mathcal{L} \) and \( \mathcal{R} \) relations on this semigroup. However, it is well-known that \( \alpha\mathcal{L}\beta \) in \( T(X) \) if and only if \( X\alpha \subseteq X\beta \) and \( \alpha\mathcal{R}\beta \) in \( T(X) \) if and only if \( \pi_\alpha = \pi_\beta \) (see [1] vol 1, Lemma 2.5 and Lemma 2.6).

**Lemma 2.** Let \( \alpha, \beta \in S(X, Y) \). Then \( \alpha = \gamma \beta \) for some \( \gamma \in S(X, Y) \) if and only if \( X\alpha \subseteq X\beta \) and \( Y\alpha \subseteq Y\beta \). Consequently, \( \alpha\mathcal{L}\beta \) if and only if \( X\alpha = X\beta \) and \( Y\alpha = Y\beta \).

**Proof.** We first note that if \( \alpha = \gamma \beta \) for some \( \gamma \in S(X, Y) \), then \( X\alpha \subseteq X\beta \) and \( Y\alpha = Y\gamma \beta = (Y\gamma)\beta \subseteq Y\beta \) since \( \gamma \in S(X, Y) \).

To prove the converse, we suppose that \( X\alpha \subseteq X\beta \) and \( Y\alpha \subseteq Y\beta \). Then \( Y(\alpha|_Y) \subseteq Y(\beta|_Y) \) where \( \alpha|_Y, \beta|_Y \in T(Y) \). Hence, by a standard result, \( \alpha|_Y = \delta(\beta|_Y) \) for some \( \delta \in T(Y) \): that is, \( y\alpha = (y\delta)\beta \) for each \( y \in Y \). Now, for each
$x \notin Y$, there exists some $x' \in X$ such that $x\alpha = x'\beta$ since $X\alpha \subseteq X\beta$. Thus for each $x \notin Y$, choose such an $x'$ and extend $\delta \in T(Y)$ to $\gamma \in T(X)$ by

$$x\gamma = \begin{cases} x\delta & \text{if } x \in Y, \\ x' & \text{if } x \notin Y. \end{cases}$$

Then $\gamma \in S(X,Y)$ and $\alpha = \gamma\beta$ as required. $\square$

We note that for any $\alpha \in S(X,Y)$, the symbol $\pi_\alpha$ will denote the composition of $X$ induced by the map $\alpha$, namely

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\},$$

and $\pi_\alpha(Y)$ will denote the subset of $\pi_\alpha$ defined by

$$\pi_\alpha(Y) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}.$$

For $\alpha, \beta \in S(X,Y)$, $A \subseteq \pi_\alpha$, and $B \subseteq \pi_\beta$, we say that $A$ refines $B$ if for each $A \in A$ there exists $B \in B$ such that $A \subseteq B$.

**Lemma 3.** Let $\alpha, \beta \in S(X,Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(X,Y)$ if and only if $\pi_\beta$ refines $\pi_\alpha$ and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Consequently, $\alpha R \beta$ if and only if $\pi_\alpha = \pi_\beta$ and $\pi_\alpha(Y) = \pi_\beta(Y)$.

**Proof.** It is clear that if $\alpha = \beta\gamma$ for some $\gamma \in S(X,Y)$, then $\pi_\beta$ refines $\pi_\alpha$ and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$.

Conversely, assume that the conditions hold. For each $x \in X\beta$, there exists $z \in X$ such that $x = z\beta$, so we define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x\alpha, & \text{if } x \in X\beta, \\ x\beta, & \text{if } x \in X \setminus X\beta. \end{cases}$$

Then $\gamma$ is well-defined since $\pi_\beta$ refines $\pi_\alpha$. Now, we prove that $\gamma \in S(X,Y)$.

For each $y \in Y$, we have $y \in X \setminus X\beta$ or $y \in X\beta \cap Y$. If $y \in X \setminus X\beta$, then $y\gamma = y\beta \in Y$ since $\beta \subseteq S(X,Y)$. If $y \in X\beta \cap Y$, then there exists $x \in X$ such that $y = x\beta$. Since $\beta \subseteq S(X,Y)$, we have $x \in y\beta^{-1} \subseteq y\alpha^{-1}$ for some $y' \in X\alpha \cap Y$. Thus $y\gamma = x\beta\gamma = x\alpha = y' \in Y$. Also, we have $x\beta\gamma = (x\beta)\gamma = x\alpha$ for all $x \in X$ by the definition of $\gamma$. $\square$

Recall that each group $H$-class of $T(X)$ is isomorphic to a permutation group $G(A)$ for some $A \subseteq X$ ([1] vol 1, Theorem 2.10). Here, for the semigroup $S(X,Y)$, the result depends on the group which is denoted by $G(A,B)$ and

$$G(A,B) = \{\rho \in G(A) : \rho|_B \in G(B)\},$$

where $B \subseteq A$ for some $A \subseteq X$ and $B \subseteq Y$.

**Theorem 3.** Let $\epsilon$ be an idempotent in $S(X,Y)$. Then the group $H$-class $H_\epsilon$ is isomorphic to $G(A,B)$. In this case, $A$ is a cross section of $\pi_\epsilon$. 

Proof. Since $\epsilon$ is an idempotent, we can write
\[
\epsilon = \begin{pmatrix} C_i & D_j \\ c_i & d_j \end{pmatrix},
\]
where $c_i \in C_i \cap Y$ and $d_j \in D_j \subseteq X \setminus Y$. Let $B = \{c_i\} \subseteq Y$ and $A = \{c_i\} \cup \{d_j\} \subseteq X$. Since $H_\epsilon = L_\epsilon \cap R_\epsilon$, we have by Lemma 2 and Lemma 3 that
\[
H_\epsilon = \left\{ \begin{pmatrix} C_i & D_j \\ c_i & d_j \delta \end{pmatrix} : \sigma \in G(B), \delta \in G(A \setminus B) \right\}.
\]
Let $\rho = \sigma \cup \delta$. Then $\rho \in G(A, B)$ and
\[
H_\epsilon = \left\{ \begin{pmatrix} C_i & D_j \\ c_i & d_j \rho \end{pmatrix} : \rho \in G(A, B) \right\}.
\]
Therefore, $H_\epsilon$ is isomorphic to $G(A, B)$ by sending $\begin{pmatrix} C_i & D_j \end{pmatrix}$ to $\rho$ where $G(A, B)$ is a subgroup of the permutation group $G(A)$. \qed

We note that when $\epsilon = id_X$, then $H_\epsilon$ the group of units of $S(X, Y)$ is isomorphic to $G(X, Y)$ and this group was shown to isomorphic to the automorphism group of $S(X, Y)$ when $|Y| > 2$ (see [7], Theorem 4.2).

Clifford and Preston in [1] vol 1, Lemma 2.8, proved that two elements of $T(X)$ are $D$-related if and only if they have the same rank, that is, the ranges of the two elements have the same cardinality. But, for $S(X, Y)$ we have the following theorem.

**Theorem 4.** Let $\alpha, \beta \in S(X, Y)$. Then $\alpha D \beta$ if and only if $|Y\alpha| = |Y\beta|$, $|X\alpha \setminus Y| = |X\beta \setminus Y|$ and $|(X\alpha \cap Y) \setminus Y\alpha| = |(X\beta \cap Y) \setminus Y\beta|$.

**Proof.** First assume that $\alpha R \gamma R \beta$ for some $\gamma \in S(X, Y)$. Then by Lemma 3, we have $\pi_\beta = \pi_\gamma$ and $\pi_\beta(Y) = \pi_\gamma(Y)$. Thus we can write
\[
\beta = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}, \quad \gamma = \begin{pmatrix} A_i & B_j & C_k \\ x_i & y_j & z_k \end{pmatrix},
\]
where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X \setminus Y; \{a_i\} \subseteq Y \setminus \{a_i\}, \{b_i\} \subseteq Y \setminus \{b_i\}$ and $\{c_k\} \subseteq X \setminus Y$. Since $X\alpha = X\gamma, Y\alpha = Y\gamma$ by Lemma 2, we must have
\[
\alpha = \begin{pmatrix} L_i & M_j & N_k \\ x_i & y_j & z_k \end{pmatrix},
\]
where $L_i \cap Y \neq \emptyset$ and $M_j, N_k \subseteq X \setminus Y$. Then $|Y\alpha| = |\{a_i\}| = |I| = |\{a_i\}| = |Y\beta|$, $|X\alpha \setminus Y| = |\{c_k\}| = |K| = |\{c_k\}| = |X\beta \setminus Y|$ and $|(X\alpha \cap Y) \setminus Y\alpha| = |\{b_j\}| = |J| = |\{b_j\}| = |(X\beta \cap Y) \setminus Y\beta|$.

Conversely, assume that the conditions hold. We can write
\[
\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}, \quad \beta = \begin{pmatrix} U_i & V_j & W_k \\ u_i & v_j & w_k \end{pmatrix},
\]
Theorem 5. \( \text{Let } \mu = \begin{pmatrix} U_i & V_j & W_k \\ a_i & b_j & c_k \end{pmatrix}, \)

thus \( \mu \in S(X,Y) \) and \( Y_\mu = \{ a_i \} = Y_\alpha, \) \( X_\mu = \{ a_i \} \cup \{ b_j \} \cup \{ c_k \} = X_\alpha. \) So \( \alpha \mathcal{L} \mu \) by Lemma 2. Also we have \( \pi_\mu = \pi_\beta \) and \( \pi_\mu(Y) = \{ U_i \} \cup \{ V_j \} = \pi_\beta(Y). \) Hence \( \mu \mathcal{R} \beta \) by Lemma 3 and therefore \( \alpha \mathcal{D} \beta. \)

Corollary 3. Let \( \alpha, \beta \in S(X,Y). \) If \( Y \) is a finite subset of \( X, \) then \( \alpha \mathcal{D} \beta \) if and only if \( |X_\alpha| = |X_\beta|, |Y_\alpha| = |Y_\beta| \) and \( |X_\alpha \cap Y| = |X_\beta \cap Y|. \)

Proof. Suppose that \( Y \) is a finite subset of \( X. \) If \( \alpha \mathcal{D} \beta, \) then by Theorem 4, we have \( |Y_\alpha| = |Y_\beta|, |X_\alpha \setminus Y| = |X_\beta \setminus Y| \) and \( |(X_\alpha \setminus Y) \setminus Y| = |(X_\beta \setminus Y) \setminus Y|. \) Since \( X_\alpha \cap Y = \alpha \mathcal{L}(X_\alpha \cap Y) \cap Y, \) it follows that \( |X_\alpha \cap Y| = |Y_\alpha| + |(X_\alpha \cap Y) \setminus Y| \) which implies that \( |(X_\alpha \cap Y) \setminus Y| = |(X_\beta \setminus Y) \setminus Y|. \) Since \( X_\alpha = (X_\alpha \cap Y) \cap (X_\alpha \setminus Y), \) we get \( |X_\alpha| = |X_\alpha \cap Y| + |X_\alpha \setminus Y| = |X_\beta \cap Y| + |X_\beta \setminus Y| = \mu \mathcal{R} \beta \) since \( |Y_\alpha| = |Y_\beta| \) is finite. Since \( |X_\alpha \cap Y| = |X_\beta \setminus Y| \) is finite and \( |X_\alpha \cap Y| + |X_\alpha \setminus Y| = |X_\alpha| = |X_\beta| = |X_\beta \cap Y| + |X_\beta \setminus Y|, \) we have \( |X_\alpha \setminus Y| = |X_\beta \setminus Y|. \) Therefore, \( \alpha \mathcal{D} \beta \) as required.

Theorem 5. Let \( \alpha, \beta \in S(X,Y). \) Then \( \alpha = \lambda \beta \mu \) for some \( \lambda, \mu \in S(X,Y) \) if and only if \( |X_\alpha| \leq |X_\beta|, |Y_\alpha| \leq |Y_\beta| \) and \( |X_\alpha \setminus Y| \leq |X_\beta \setminus Y|. \) Consequently, \( \alpha \mathcal{J} \beta \) if and only if \( |X_\alpha| = |X_\beta|, |Y_\alpha| = |Y_\beta| \) and \( |X_\alpha \setminus Y| = |X_\beta \setminus Y|. \)

Proof. Assume that \( \alpha = \lambda \beta \mu \) for some \( \lambda, \mu \in S(X,Y). \) Then

\[
|X_\alpha| = |X_\lambda \beta \mu| = |(X_\lambda \beta) \mu| \leq |X_\lambda \beta| = |(X_\lambda) \beta| \leq |X_\beta|,
\]

\[
|Y_\alpha| = |Y_\lambda \beta \mu| = |(Y_\lambda \beta) \mu| \leq |Y_\lambda \beta| = |(Y_\lambda) \beta| \leq |Y_\beta|,
\]

and

\[
|X_\alpha \setminus Y| = |X_\lambda \beta \mu \setminus Y| = |(X_\lambda \beta) \mu \setminus Y| \leq |X_\beta \setminus Y|,
\]

Conversely, assume that the conditions hold and write

\[
\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix},
\]
where $A_i \cap Y \neq \emptyset$; $B_j, C_k \subseteq X \setminus Y$; $\{a_i\} \subseteq Y, \{b_j\} \subseteq Y \setminus \{a_i\}, \{c_k\} \subseteq X \setminus Y$. By the assumption we can write

$$\beta = \left( \begin{array}{cccc}
U_i & V_m & W_n & W_k \\
u_i & u_l & v_m & w_k
\end{array} \right),$$

where $U_i \cap Y \neq \emptyset \neq U_l \cap Y; V_m, W_n, W_k \subseteq X \setminus Y; \{u_i, u_l\} \subseteq Y, \{v_m\} \subseteq Y \setminus \{u_i, u_l\}, \{w_n, w_k\} \subseteq X \setminus Y$ and $|I| + |J| + |K| \leq |I| + |L| + |M| + |N| + |K|$. We consider in two cases:

**Case 1:** $|J| \leq |L| + |M| + |N|$. Let $I \cup M \cup N = P \cup Q$ where $|P| = |J|$. Then we can write $\{U_i\} \cup \{V_m\} \cup \{W_n\} = \{S_p\} \cup \{S_q\}$ and rewrite $\beta$ as follows:

$$\beta = \left( \begin{array}{cccc}
U_i & S_p & V_m & W_n & W_k \\
u_i & s_p & u_l & v_m & w_k
\end{array} \right).$$

Since $|J| = |P|$, there is a bijection $\varphi : J \rightarrow P$. Now define

$$\lambda = \left( \begin{array}{cccc}
A_i & B_j & C_k \\
x_i & y_j & z_k
\end{array} \right),$$

where $x_i \in U_i \cap Y, y_j \varphi \in S_j \varphi, z_k \in W_k$. So $\lambda \in S(X, Y)$. Choose $i_0 \in I$ and let $I' = I \setminus \{i_0\}$. Then define

$$\mu = \left( \begin{array}{cccc}
u_i & s_j & w_k & X \setminus \{u_i, s_j, w_k\} \\
a_i & b_j & c_k & a_{i_0}
\end{array} \right).$$

So $\mu \in S(X, Y)$ and $\alpha = \lambda \beta \mu$.

**Case 2:** $|J| > |L| + |M| + |N|$. Then $X \beta$ is infinite (for if $X \beta$ is finite, then $|X\alpha| = |I| + |J| + |K| > |I| + |L| + |M| + |N| + |K| = |X \beta|$ which is a contradiction). Hence $|J| \leq |I| or |J| \leq |K|$ are infinite cardinals. If $|J| \leq |I|$ is an infinite cardinal, then write $I = P \cup Q$ where $|P| = |J|, |Q| = |J|$. Thus we can write $\{U_i\} = \{S_p\} \cup \{S_q\}$ and rewrite $\beta$ as follows:

$$\beta = \left( \begin{array}{cccc}
S_p & S_q & U_i & V_m & W_n & W_k \\
s_p & s_q & u_i & v_m & w_n & w_k
\end{array} \right).$$

Since $|I| = |P|$ and $|J| = |Q|$, there are bijections $\varphi : I \rightarrow P$ and $\psi : J \rightarrow Q$. Then define $\lambda$ and $\mu$ as follows:

$$\lambda = \left( \begin{array}{cccc}
A_i & B_j & C_k \\
x_i & y_j & z_k
\end{array} \right),$$

where $x_{i \varphi} \in S_i \varphi \cap Y, y_{j \psi} \in S_j \psi, z_k \in W_k$, and

$$\mu = \left( \begin{array}{cccc}
sl_i & s_j & w_k & X \setminus \{s_i \varphi, s_j \psi, w_k\} \\
a_i & b_j & c_k & a_{i_0}
\end{array} \right),$$

where $I' = I \setminus \{i_0\}$ for some fixed $i_0 \in I$. So, we see that $\lambda, \mu \in S(X, Y)$ and $\alpha = \lambda \beta \mu$. For the case $|J| \leq |K|$ is an infinite cardinal, we write $K = G \cup H$.
where $|G| = |J|$, $|H| = |K|$. Write $\{W_k\} = \{T_g\} \cup \{T_h\}$ and rewrite $\beta$ as follows:

$$\beta = \begin{pmatrix} u_1 & u_t & v_m & w_n & T_g & T_h \\ u_t & u_1 & v_m & w_n & T_g & T_h \end{pmatrix}. $$

As above, we can define $\lambda, \mu \in S(X, Y)$ such that $\alpha = \lambda \beta \mu$. \(\Box\)

The following example shows that in general $D \neq J$ on $S(X, Y)$.

**Example 2.** Let $X = N$ and $Y$ the set of positive even integers. Then we define

$$\alpha = \left( \frac{n}{2n} \right)_{n \in \mathbb{N}} \quad \text{and} \quad \beta = \left( \frac{2n}{4n} \frac{X \setminus Y}{2} \right)_{n \in \mathbb{N}}. $$

Hence $\alpha, \beta \in S(X, Y)$ and $|X\alpha| = N_0 = |X\beta|, |Y\alpha| = N_0 = |Y\beta|, |X\alpha \setminus Y| = 0 = |X\beta \setminus Y|$, so $\alpha J \beta$. Since $|(X\alpha \cap Y) \setminus Y\alpha| = N_0 \neq 1 = |(X\beta \cap Y) \setminus Y\beta|$, we have $\alpha$ and $\beta$ are not $D$-related on $S(X, Y)$.

Even $Y$ is a finite proper subset of $X$, we still have $D \neq J$ on $S(X, Y)$.

**Example 3.** Let $X = N$ and $Y = \{1, 2, 3, 4\}$. Then we define

$$\alpha = \left( \frac{\{1, 2\}}{3} \frac{\{3, 4\}}{1} \frac{n + 4}{n + 4} \right)_{n \in \mathbb{N}} \quad \text{and} \quad \beta = \left( \frac{\{1, 2, 4\}}{1} \frac{\{5, 6\}}{2} \frac{n + 6}{n + 4} \right)_{n \in \mathbb{N}}. $$

Hence $\alpha, \beta \in S(X, Y)$ and $|X\alpha| = N_0 = |X\beta|, |Y\alpha| = 2 = |Y\beta|, |X\alpha \setminus Y| = N_0 = |X\beta \setminus Y|$, so $\alpha J \beta$. Since $|X\alpha \cap Y| = 2$ but $|X\beta \cap Y| = 3$, we have $\alpha$ and $\beta$ are not $D$-related on $S(X, Y)$ by Corollary 3.

**Theorem 6.** $D = J$ on $S(X, Y)$ if and only if $X$ is a finite set or $X = Y$ or $|Y| = 1$.

*Proof.* If $X$ is a finite set, then by [3], Proposition 2.1.4 we have $D = J$. If $X = Y$, then $S(X, Y) = T(X)$ and thus $D = J$ by [1], Theorem 2.9(i). If $|Y| = 1$, then $S(X, Y) = E$. Let $\alpha, \beta \in S(X, Y)$ be such that $\alpha J \beta$. So, $|X\alpha| = |X\beta|, |Y\alpha| = |Y\beta|$ and $|X\alpha \setminus Y| = |X\beta \setminus Y|$. Since $\alpha, \beta \in E$, we have $|(X\alpha \cap Y) \setminus Y\alpha| = |Y\alpha \setminus Y\alpha| = 0 = |Y\beta \setminus Y\beta| = |(X\beta \cap Y) \setminus Y\beta|$ and hence $\alpha D \beta$. Thus $D = J$.

Conversely, assume that $D = J$ on $S(X, Y)$, and suppose on contrary that $X$ is an infinite set, $Y \subsetneq X$ and $|Y| \geq 2$. Let $a, b$ be two distinct elements in $Y$ and $c \in X \setminus Y$. We consider in two cases:

**Case 1:** $Y$ is a finite set. Then $|X\alpha| = |X|$ and define $\alpha, \beta \in S(X, Y)$ as follows:

$$\alpha = \begin{pmatrix} Y & c & x \\ a & b & x \end{pmatrix}_{x \in X \setminus (Y \cup \{c\})} \quad \text{and} \quad \beta = \begin{pmatrix} Y & x \\ a & x \end{pmatrix}_{x \in X \setminus Y}. $$

Thus $|X\alpha| = |X\setminus Y| = |X\beta|, |Y\alpha| = 1 = |Y\beta|$ and $|X\alpha \setminus Y| = |X\setminus Y| = |X\beta \setminus Y|$, we get $\alpha J \beta$. But, $|X\alpha \cap Y| = 2 \neq 1 = |X\beta \cap Y|$, so $\alpha$ and $\beta$ are not $D$-related by Corollary 3 and this leads to a contradiction.
Case 2: \(Y\) is an infinite set. Then define \(\alpha, \beta \in S(X, Y)\) as follows:
\[
\alpha = \left( \frac{x}{x} \frac{x}{a} \frac{\{a, b\} \ X \setminus Y}{x} \frac{x}{b} \right)_{x \in Y \setminus \{a, b\}} \quad \text{and} \quad \beta = \left( \frac{x}{x} \frac{x}{a} \frac{(X \setminus Y) \cup \{a\}}{x} \right)_{x \in Y \setminus \{a\}}.
\]
Since \(|X\alpha| = |Y| = |X\beta|, |Y\alpha| = |Y| = |Y\beta|\) and \(|X\alpha \setminus Y| = 0 = |X\beta \setminus Y|\), we get \(\alpha \not\sim \beta\). But, \((X \alpha \cap Y) \setminus Y\alpha| \neq 0 = (X \beta \cap Y) \setminus Y\beta|\), so \(\alpha\) and \(\beta\) are not \(\mathcal{D}\)-related which is a contradiction. \(\square\)

As a direct consequence of Theorem 6, we have the following corollary.

**Corollary 4.** If \(X\) is an infinite set and \(Y\) is a finite subset of \(X\), then \(\mathcal{D} = \mathcal{J}\) on \(S(X, Y)\) if and only if \(|Y| = 1\).

4. **Ideals of \(S(X, Y)\)**

Let \(p\) be any cardinal number and let
\[
p' = \min\{q : q > p\}.
\]
Note that \(p'\) always exists since the cardinals are well-ordered and when \(p\) is finite we have \(p' = p + 1 = \) the successor of \(p\). As shown by Malcev [5], the ideals of \(T(X)\) for any set \(X\) are precisely the sets:
\[
T_r = \{\alpha \in T(X) : |X\alpha| < r\},
\]
where \(2 \leq r \leq |X'|\) (see also [1] vol 2, Theorem 10.59).

To describe ideals of \(S(X, Y)\) for any set \(X\) and any nonempty subset \(Y\) of \(X\), we let \(|X| = a, |Y| = b\) and \(|X \setminus Y| = c\). In addition, for each cardinals \(r, s, t\) such that \(2 \leq r \leq a', 2 \leq s \leq b'\) and \(1 \leq t \leq c'\), define
\[
S(r, s, t) = \{\alpha \in S(X, Y) : |X\alpha| < r, |Y\alpha| < s, \text{ and } |X\alpha \setminus Y| < t\}.
\]
Then if \(r = a'\) or \(s = b'\) or \(t = c'\), the set \(S(r, s, t)\) can be deduced in a simple form:
\[
\begin{align*}
S(a', s, t) &= \{\alpha \in S(X, Y) : |Y\alpha| < s, |X\alpha \setminus Y| < t\}, \\
S(r, b', t) &= \{\alpha \in S(X, Y) : |X\alpha| < r, |X\alpha \setminus Y| < t\}, \\
S(r, s, c') &= \{\alpha \in S(X, Y) : |X\alpha| < r, |Y\alpha| < s\}, \\
S(a', b', t) &= \{\alpha \in S(X, Y) : |X\alpha \setminus Y| < t\}, \\
S(r, b', c') &= \{\alpha \in S(X, Y) : |X\alpha| < r\}, \\
S(a', s, c') &= \{\alpha \in S(X, Y) : |Y\alpha| < s\}, \\
\text{and} \quad S(a', b', c') &= S(X, Y).
\end{align*}
\]
We observe that: if \(X = Y\), then \(|X| = a = |Y|\) and \(|X \setminus Y| = 0\), thus \(S(r, r, 1) = \{\alpha \in S(X, Y) : |X\alpha| < r\} = \{\alpha \in T(X) : |X\alpha| < r\}\) which is an ideal of \(T(X)\).

**Theorem 7.** The set \(S(r, s, t)\) is an ideal of \(S(X, Y)\).
Proof. Let \( \alpha \in S(r, s, t) \) and \( \lambda, \mu \in S(X, Y) \). Then \( |X\alpha| < r, |Y\alpha| < s \) and \( |X\alpha\setminus Y| < t \). Thus by using the same proof as given in Theorem 5, we get \( |X(\lambda\alpha\mu)| \leq |X\alpha| < r, |Y(\lambda\alpha\mu)| \leq |Y\alpha| < s \) and \( |X(\lambda\alpha\mu)\setminus Y| \leq |X\alpha\setminus Y| < t \). Hence \( \lambda\alpha\mu \in S(r, s, t) \). Therefore, \( S(r, s, t) \) is an ideal of \( S(X, Y) \). \( \Box \)

We note that if \( r \leq u, s \leq v \) and \( t \leq w \), then \( S(r, s, t) \subseteq S(u, v, w) \). The following example shows that there is an ideal in \( S(X, Y) \) which is not of the form \( S(r, s, t) \) and the set of ideals of \( S(X, Y) \) does not form a chain under the set inclusion.

**Example 4.** Let \( X = \{1, 2, 3, 4\} \) and \( Y = \{1, 2\} \). Then \( |X| = 4, |Y| = 2 \) and \( |X \setminus Y| = 2 \). Since \( S(3, 3, 1) \) and \( S(4, 2, 2) \) are ideals of \( S(X, Y) \), we have \( S(3, 3, 1) \cup S(4, 2, 2) \) is also an ideal of \( S(X, Y) \). Suppose that \( S(3, 3, 1) \cup S(4, 2, 2) = S(\ell, m, n) \) for some \( 2 \leq \ell \leq 5, 2 \leq m \leq 3 \) and \( 1 \leq n \leq 3 \). If \( \ell < 4 \) or \( m < 2 \), then there is \( r = \binom{\{1, 2\}}{2} \) \( \in S(4, 2, 2) \setminus S(\ell, m, n) \). If \( m < 3 \) then there is \( r = \binom{\{1, 2\}}{4} \) \( \in S(3, 3, 1) \setminus S(\ell, m, n) \). Both cases contradict our supposition. So \( \ell \geq 4, m \geq 3 \) and \( n \geq 2 \). Consider \( \delta = \binom{\{1, 2\}}{3, 4} \in S(4, 3, 2) \), but \( \delta \notin S(3, 3, 1) \cup S(4, 2, 2) \), so \( S(3, 3, 1) \cup S(4, 2, 2) \neq S(r, s, t) \) for all \( r \geq 4, s \geq 3 \) and \( t \geq 2 \). Since \( \alpha \in S(4, 2, 2) \), \( S(3, 3, 1) \) and \( \beta \in S(3, 3, 1) \), \( S(4, 2, 2) \), we conclude that the set of ideals of \( S(X, Y) \) does not form a chain.

To obtain ideals of \( S(X, Y) \) we need the following notation. Let \( Z \) be a nonempty subset of \( S(X, Y) \). Define

\[
K(Z) = \{\alpha \in S(X, Y) : |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \text{ and } |X\alpha \setminus Y| \leq |X\beta \setminus Y| \text{ for some } \beta \in Z\}.
\]

Then we see that \( Z \subseteq K(Z) \) and \( Z_1 \subseteq Z_2 \) implies \( K(Z_1) \subseteq K(Z_2) \).

**Theorem 8.** The ideals of \( S(X, Y) \) are precisely the set \( K(Z) \) for some nonempty subset \( Z \) of \( S(X, Y) \).

**Proof.** Let \( I \) be an ideal of \( S(X, Y) \). We prove that \( I = K(I) \). If \( \alpha \in K(I) \), then \( |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \) and \( |X\alpha \setminus Y| \leq |X\beta \setminus Y| \) for some \( \beta \in I \) and thus by Theorem 5 we have \( \alpha = \lambda\beta\mu \) for some \( \lambda, \mu \in S(X, Y) \). Since \( \beta \in I \) is an ideal of \( S(X, Y) \), it follows that \( \alpha = \lambda\beta\mu \in I \) and that \( K(I) \subseteq I \). Usually, we have \( I \subseteq K(I) \). Therefore, \( I = K(I) \).

Conversely, we prove that \( K(Z) \) is an ideal of \( S(X, Y) \). Let \( \alpha \in K(Z) \) and \( \lambda, \mu \in S(X, Y) \). Then \( |X\alpha| \leq |X\beta|, |Y\alpha| \leq |Y\beta| \) and \( |X\alpha \setminus Y| \leq |X\beta \setminus Y| \) for some \( \beta \in Z \). Like before, we have \( |X(\lambda\alpha\mu)| \leq |X\alpha|, |Y(\lambda\alpha\mu)| \leq |Y\alpha| \) and \( |X(\lambda\alpha\mu) \setminus Y| \leq |X\alpha \setminus Y| \). Thus \( |X(\lambda\alpha\mu)| \leq |X\beta|, |Y(\lambda\alpha\mu)| \leq |Y\beta| \) and \( |X(\lambda\alpha\mu) \setminus Y| \leq |X\beta \setminus Y| \). Hence \( \lambda\alpha\mu \in K(Z) \) and therefore \( K(Z) \) is an ideal of \( S(X, Y) \). \( \Box \)

The following result was first proved by Malcev [5] in 1952.
Corollary 5. The ideals of $T(X)$ are precisely the set $T(r) = \{\alpha \in T(X) : |X\alpha| < r\}$, where $2 \leq r \leq |X'|$.

Proof. By taking $Y = X$ in Theorem 8, we see that the ideals of $S(X, X) = T(X)$ are precisely the set $K(Z)$ for some nonempty subset $Z$ of $T(X)$. Let $r$ be the least cardinal of $A = \{s : s > |X\beta| \text{ for all } \beta \in Z\}$ ($A$ is nonempty since $|X'| \in A$). Then for each $\alpha \in K(Z)$, there is $\beta \in Z$ such that $|X\alpha| \leq |X\beta| < r$ and thus $K(Z) \subseteq T(r)$. Conversely, suppose that $\alpha \notin K(Z)$, then $|X\alpha| > |X\beta|$ for all $\beta \in Z$. Thus $|X\alpha| \in A$ and hence $|X\alpha| \geq r$ since $r$ is the least cardinal of $A$, that means $\alpha \notin T(r)$. So, $T(r) \subseteq K(Z)$ and therefore $K(Z) = T(r)$. □

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